Reconfigurability and unified kinematics modeling of a 3rTPS metamorphic parallel mechanism with perpendicular constraint screws

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This paper investigates reconfigurability and unified analytical kinematics analysis of a new 3rTPS metamorphic parallel mechanism consisting of three reconfigurable rTPS limbs in perpendicular base planes. Constraint screw systems show that in one phase the rTPS limb has no constraint to the platform and in the other phase obtained by altering the reconfigurable Hooke joint, it provides a constraint force. The two phases of the limb qualify the 3rTPS metamorphic parallel mechanism to have four topologies with ability of mobility change among 3R (three translations), 3RT (three rotations one translation), 3R2T and mobility 6. By considering the difference of the two phases of the limb, a unified kinematics modeling is proposed based on the actuation scheme analysis by taking one phase as a special case of the other. Following this, a unified kinematics modeling of the 3rTPS metamorphic parallel mechanism is obtained by covering all its four topologies. Both inverse and forward kinematics analysis are solved analytically and numerical examples confirm these theoretical results.

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1. Introduction

Since Gough [1] used a parallel mechanism for the tire test and Stewart [2] further described it for the flight simulator, parallel mechanism demonstrated great advantages in industry assembly [3, 4], manufacturing [5], force transducer [6], rehabilitation [7] and robotics surgery [8] due to its high load-carrying capacity, good positioning accuracy and low inertia [9] with respect to the serial counterparts. Motivated by the applications, various researches have been taken to investigate the basic theories of parallel mechanisms to solve the kinematics [10, 11], dynamics [12], workspace [13], optimization [14] and synthesis [15, 16] of novel structures. With the requirements of rapid customization and diverse environment changing [17], parallel mechanisms that have reconfigurability to adapt to different scenarios have attracted much attention in recent mechanism research. Based on this, metamorphic parallel mechanisms (MPMs) [18], which are a class of mechanisms that possess adaptability and reconfigurability to change permanent finite mobility based on topological structure change, were introduced. Metamorphic parallel mechanisms keep the advantages of traditional parallel mechanisms but with ability of reconfiguring and mobility change.

Research of mechanisms that have reconfigurability started in 1990s when Wohlhart [19] presented kinematotropic linkages with variable position parameters and Dai and Rees Jones [20] proposed metamorphic mechanisms in the study of decorative carton folds and reconfigurable packaging. Following this, many contributions have been made on this topic. Parise et al. [21] presented new ortho-planar metamorphic mechanisms. Liu and Yang [22] investigated three metamorphic ways of changing the topological structures of a mechanism. Dai and Rees [23] presented the topology of various mobility configurations of metamorphic mechanisms by using matrix operations while Yan and Kuo [24, 25] investigated variable topologies of kinematic joints and presented the topological representation of joints. Based on single-loop and multi-loop kinematotropic chains, Fanghella and Galletti [26] developed kinematotropic parallel mechanisms with variable motions. Refaat and Herve et al. [27] proposed a parallel mechanism with changed motion for machine tool applications. A family of parallel mechanisms that have multiple operation modes with pure translation and pure rotation were presented by Kong et al. [28]. Later, Dai and Wang [29] presented a metamorphic multifingered hand with an articulated palm while Zhang and Dai [30] developed a methodology for synthesis and configuration design of metamorphic mechanisms based on biological modeling and genetic evolution. Recently, Zhang et al. [31] proposed a metamorphic parallel mechanism with the ability of performing phase change and orientation switch by introducing a metamorphic kinematic pair.
The new metamorphic parallel mechanism proposed in this paper is based on a reconfigurable rTPS limb with two topology phases stemming from a reconfigurable Hooke (rT) joint [32]. Based on this rT joint, various metamorphic parallel mechanisms [1,33] have been presented with a general construction method introduced using screw theory [34]. As screws [35] naturally present the relations of geometric constraints among the joints and limbs in parallel mechanisms, they show great advantage in describing the topology change of metamorphic parallel mechanisms as illustrated in this paper.

Since many novel metamorphic parallel mechanisms have been proposed, further analysis, such as kinematics, dynamics and optimization, has not been identified on them. Especially, when considering property changes and variable topologies of a metamorphic parallel mechanism, unified kinematics, dynamics and other interpreting methods are required. A unified kinematics modeling is for the first time investigated in this paper following the topology change of the 3rTPS metamorphic parallel mechanism.

In the following, the paper is arranged as: Section 2 introduces the two phases of the rTPS limb with its geometric constraint analysis in screw theory; following this, mobility change of the 3rTPS metamorphic parallel mechanism is demonstrated in different topologies in Section 3 and possible actuation schemes are discussed in Section 4; based on these, a unified kinematics modeling of the 3rTPS metamorphic parallel mechanism is proposed by covering all its topologies in Section 5 with numerical examples illustrating the validity; conclusions make up Section 6.

2. Two phases of the reconfigurable rTPS limb

As in Fig. 1, the reconfigurable rTPS limb consists of a reconfigurable Hooke (rT) joint, a prismatic joint and a spherical joint. The reconfigurability of this limb stems from the configuration change of the rT joint which has two rotational degrees of freedom (DOFs) about two perpendicularly intersecting rotational axes (radial axis and bracket axis) as in Fig. 1. A grooved ring is used to house the radial axis and make it have the ability of altering its direction by rotating and fixing freely along the groove. This allows the radial rotation axis change with respect to the limb, resulting in two typical phases of the rTPS limb as in Fig. 1. In Fig. 1(a), the radial axis is perpendicular to the limb (prismatic joint) which is denoted as phase (rT)PS. The radial axis is collinear with the limb (prismatic joint) passing through the spherical joint center as in Fig. 1(b) and the limb phase is expressed as (rT)2PS.

Set a limb coordinate system 1o1x1y1z at the rT joint center with 1x axis collinear with the bracket axis and 1y axis perpendicular to the bracket surface as in Fig. 1(a), the twist system of the (rT)1PS limb is given as

\[
\{\mathbf{S}_1\} = \begin{bmatrix}
1s_{11} &=& 1 & 0 & 0 & 0 & 0 \\
1s_{12} &=& 0 & -sz & cz & 0 & 0 \\
1s_{13} &=& 0 & 0 & -s\beta & c\beta cx & c\beta sz \\
1s_{14} &=& 0 & sz & 0 & lsz & ls\beta cx & -ls\beta cz \\
1s_{15} &=& 1 & 0 & 0 & 0 & lcz & ls\beta cx & -ls\beta sz \\
1s_{16} &=& 0 & -sz & cz & l & 0 & 0 & 0
\end{bmatrix}
\]

(1)

where the first two twists are generated from the rT joint, the third is generated from the prismatic joint, and the last three are generated from the spherical joint. \(\beta\) is the angle between the limb (1s13) and its projection on plane 1y1o1z passing through rT joint center and perpendicular to the bracket axis (1x), \(\alpha\) is the angle between axis 1y and the limb projection on plane 1y1o1z. \(l\) is the distance between the rT joint center and the spherical joint center. Labels sz and sz denote cosine(\(\alpha\)) and sine(\(\alpha\)), respectively.

In the twist notation \(\mathbf{S}_n\), the first subscript \(i\) denotes the limb number, the second subscript \(j\) denotes the joint number within the limb and the leading superscript indicates the local frame.

The six screws in (1) form a six-system [36] and there is no reciprocal screw, showing that the (rT)1PS limb has six degrees of freedom (DOFs) and does not supply constraint to the platform connected to it.

For the (rT)2PS limb as in Fig. 1(b), the radial axis of the rT joint is collinear with the prismatic joint passing through the spherical joint center. Thus, the spherical joint center \(A\) cannot move out of the plane 1y1o1z. Based on the coordinate system in Fig. 1(b), the twist system of the (rT)2PS limb can be given as

\[
\{\mathbf{S}_1\} = \begin{bmatrix}
1s_{11} &=& 1 & 0 & 0 & 0 & 0 \\
1s_{12} &=& 0 & -cz & sz & 0 & 0 \\
1s_{13} &=& 0 & 0 & 0 & cz & sz \\
1s_{14} &=& 0 & cz & sz & 0 & 0 \\
1s_{15} &=& 1 & 0 & 0 & 0 & lcz & 0 \\
1s_{16} &=& 0 & -sz & cz & l & 0 & 0
\end{bmatrix}
\]

(2)

It is clear that twists 1S12 and 1S14 are the same and the six twists form a five-system [36]. Thus, there is one reciprocal screw to (2) in the limb constraint system as

\[
\{\mathbf{S}_2\} = {\mathbf{S}_1}^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -cz & sz & 0 & 0 \\
0 & cz & sz & 0 & 0 \\
1 & 0 & 0 & 0 & lcz \\
0 & -sz & cz & l & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(3)

This gives a constraint force acting along a line passing through the spherical joint center with a direction parallel to the bracket axis of the rT joint. Thus, the (rT)2PS limb has five DOFs, one less than the (rT)1PS phase.

When constructing parallel mechanisms with the rTPS limbs, the mechanisms will have ability of mobility change by altering the rTPS limbs into these two phases. In the following, the (rT)2PS limb phases will be used by connecting the rT joints to the base with perpendicular bracket axes and the spherical joints to the moving platform. Then altering the limbs into the (rT)1PS phase will generate new mechanism topologies with mobility change.

3. 3rTPS MPM with perpendicular constraint screws

3.1. Mobility analysis

The 3rTPS metamorphic parallel mechanism is shown in Fig. 2 with all limbs in phase (rT)2PS. The three limbs are arranged symmetrically on three perpendicular planes with the bracket axes perpendicular to the planes as in Fig. 2.
The coordinate system $B_i$ through the $rT$ joint center is spherical joint center in limb $A_i$.

Constraint screws has mobility three with pure rotations, which the axes of the coordinate system $oxyz$ which represent three rotational DOFs with directions parallel to $S_i$ joint center $B_i$ in limb 3 as in Fig. 2. Then, z-axis passes through $rT$ joint center $B_i$ in limb 1. Let $a_i$ and $b_i$ denote the vectors of points $A_i$ and $B_i$ in the coordinate system $oxyz$, and $l_i$ be the limb length between the spherical joint center $A_i$ and the $rT$ joint center $B_i$.

Based on the description in Fig. 2, the parameters expressed in the coordinate system $oxyz$ can be given as

\[
\begin{align*}
    b_1 &= (0,0,r_0), a_1 = b_1 + l_1(\sin z_1, 0, -\cos z_1) \\
    b_2 &= (r_2,0,0), a_2 = b_2 + l_2(-\cos z_2,\sin z_2,0) \\
    b_3 &= (0,r_3,0), a_3 = b_3 + l_3(0,-\cos z_3,\sin z_3)
\end{align*}
\]

where $z_i$ is the angle between limb $i$ and the line $oB_i$ passing through the $rT$ joint center $B_i$ and the coordinate system origin $o$.

From (3), it can be seen that $(rT)_2PS$ limb supplies one constraint force to the platform with the direction parallel to its bracket axis. Thus, the constraint system of the 3D $(rT)_2PS$ parallel mechanism in Fig. 2 can be given as

\[
S' = \begin{pmatrix}
    [0,1,0] & a_1 \times [0,1,0] \\
    [0,0,1] & a_2 \times [0,0,1] \\
    [1,0,0] & a_3 \times [1,0,0]
\end{pmatrix}
\]

which are three constraint forces perpendicular to each other.

By taking reciprocal screws to (5), motion screws of the mechanism can be obtained as

\[
S_m = \begin{pmatrix}
    1 & 0 & 0 & r_2-l_1c_{z_1} & -l_2s_{z_2} \\
    0 & 1 & 0 & l_2s_{z_2} & l_2c_{z_2} - r_3 \\
    0 & 0 & 1 & r_3-l_1c_{z_1} & -l_2s_{z_2}
\end{pmatrix}
\]

which represent three rotational DOFs with directions parallel to the axes of the coordinate system $oxyz$.

Thus, the 3D $(rT)_2PS$ parallel mechanism with perpendicular constraint screws has mobility three with pure rotations, which is similar to the pyramid parallel mechanism [37] consisting of three $RPS$ limbs in three perpendicular constraint planes.

### 3.2. Topology change of the 3$rTPS$ MPM

Altering the $(rT)_2PS$ limbs in the previous 3D $(rT)_2PS$ parallel mechanism into the phase $(rT)_1PS$ will result in various new mechanism topologies with increased mobility. After changing the phase of limb 1, the 3D $(rT)_2PS$ parallel mechanism becomes the topology in Fig. 3 with name of 2D $(rT)_2PS–1(rT)_1PS$.

Since the $(rT)_1PS$ limb does not supply any constraint on the platform, the first constraint screw in (5) vanishes and the other two constraint forces in (5) are left on the platform as in Fig. 3. The reciprocity of this two-screw system gives one more motion screw than those three in (6) as

\[
S_{m_1} = \begin{pmatrix}
    0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

which is a translation along y-axis.

Thus, the 3D $(rT)_2PS–1(rT)_1PS$ parallel mechanism has four DOFs with one translation and three rotations, which is one more DOF than the 3D $(rT)_2PS$ in Fig. 2. Hence, with the topology change from 3D $(rT)_2PS$ to 2D $(rT)_2PS–1(rT)_1PS$, the mobility of the mechanism increases to one.

When further changing the phase of limb 2 to $(rT)_1PS$, the 2D $(rT)_2PS–1(rT)_1PS$ topology in Fig. 3 changes to the 1D $(rT)_2PS–2(rT)_1PS$ in Fig. 4(b). Based on the platform constraint system in (5), only the third constraint screw is left and platform motion screw system has one more translation screw along z-axis than...
those in (6) and (7). Thus, the $1(rT)_2PS-2(rT)_1PS$ parallel mechanism has five DOFs with two translations and three rotations, indicating that one more mobility gained after changing the topology from $2(rT)_2PS-1(rT)_1PS$ to $1(rT)_2PS-2(rT)_1PS$.

Finally, when changing the third limb to phase $(rT)_1PS$, the mechanism becomes another topology $3(rT)_1PS$ as shown in Fig. 4(b) with no constraint on the platform. Thus, the mechanism changes to the topology with mobility 6.

The foregoing analysis shows that when changing phases of limb 1, limb 2 and limb 3 one after another, the $3(rT)_2PS$ parallel mechanism changes to three other topologies with mobility change from 3 to 4, to 5 and then to 6. However, this only shows one procedure of topology change of the $3rTPS$ parallel mechanism. When changing the limb phases in different orders and with different numbers at the same time, the four topologies can be altered into any other. The rule is that there is the translation along y-axis when limb 1 is in phase $(rT)_1PS$ and it is constrained when limb 1 is in phase $(rT)_2PS$. Similarly, limb 2 and limb 3 correspond to the translation along z and x-axes, respectively. The combination of the phases of the three limbs contributes to different mechanism topologies with variable mobility.

Based on this, a mobility change map can be illustrated in Fig. 5, in which $3R1T_a$ means three rotations with one translation along k-axis, $3R2T_{xy}$ means three rotations with two translations along k and i-axes, and $3R3T$ represent full mobility 6. The double-arrow lines between any two mobility types indicate that any two of them can be changed into each other. For example, change from 3R to 3R2T can be realized by altering phases of limb 1 and limb 3 from $(rT)_3PS$ to $(rT)_1PS$ at the same time based on the topology $3(rT)_2PS$ in Fig. 2. The obtained $1(rT)_2PS-2(rT)_1PS$ topology has three rotations and two translations along x and y-axes.

4. Actuation scheme for the reconfigurable topologies

Actuation inputs of a parallel mechanism work with limb constraints to determine the platform position and orientation. In the $(rT)_2PS$ limb, the rotation about the bracket axis or the translation along the prismatic joint can be used as inputs. Based on Fig. 1(b) and the limb motion screws in (2), these two inputs can result in constraint screws as

$$\begin{align*}
S_{rTba} &= \begin{bmatrix} 0 & -sz & cz & 0 & 0 \\ 0 & cz & sz & 0 & 0 \end{bmatrix} \\
S_{rTy} &= \begin{bmatrix} 0 & -sz & cz & 0 & 0 \\ 0 & cz & sz & 0 & 0 \end{bmatrix}
\end{align*}$$

where $S_{rTba}$, a constraint force passing through the spherical joint center and perpendicular to both the bracket axis and the limb, is the input constraint from the bracket axis rotation. $S_{rTy}$, a constraint force along the limb, is the input constraint from the prismatic joint.

When altering the $(rT)_2PS$ limb to phase $(rT)_1PS$, one more actuation choice appears as the rotation about the radial axis with an input constraint force passing through the spherical joint center and parallel to the bracket axis as described by (3).

The rule for a satisfied actuation scheme is that the input constraint screws form a six-system with the platform constraint screw system. Based on this and the above analysis, the actuation scheme of the 3rTPS metarobotic parallel mechanism can be concluded: (a) $3+n$ actuation inputs are needed for the topology $3(rT)_2PS-n(rT)_1PS$; (b) any one of the three inputs (rotation about the bracket axis and translation of limb $(rT)_2PS$, rotation about the radial axis of limb $(rT)_1PS$) can be used as actuation; and (c) at least one of them should be from each $(rT)_1PS$ limb if there is.

To demonstrate the rule, the $1(rT)_2PS-2(rT)_1PS$ parallel mechanism in Fig. 4(a) can be taken as an example. Based on the scheme concluded, $3+2=5$ actuation inputs are needed for this topology, which can be chosen from the eight choices (the $(rT)_2PS$ limb has two, each $(rT)_1PS$ limb has three) and two of them should be from the two $(rT)_1PS$ limbs. Taking the three translations of the prismatic joints in the three limbs and the rotations about the radial axes of the two $(rT)_1PS$ limbs as an example, the actuation input constraint screw system can be given as

$$S_a = \begin{bmatrix}
S_{1rTba} &= \begin{bmatrix} u_1 & b_1 	imes u_1 \\ u_2 & b_2 	imes u_2 \end{bmatrix} \\
S_{2rTba} &= \begin{bmatrix} u_1 & a_1 \times u_0 \\ u_2 & a_2 \times u_0 \end{bmatrix} \\
S_{3rTy} &= \begin{bmatrix} u_3 & b_3 \times u_3 \\ 0 & 0 \end{bmatrix}
\end{align*}$$

where $u_i$ is the unit vector along limb $i$ and can be expressed in (11), $x$ and $y$ are shown in Fig. 1. $u_0$ is the unit vector perpendicular to both limb $i$ and its radial axis.

The five input force screws with the third geometric constraint screw in (5) form a six-system, showing that the selected actuation inputs can fix the required position and orientation of the platform. This also shows the validity for the unified kinematics modeling by choosing the rotation about the radial axis as input in the following section.

5. Unified kinematics modeling and displacement analysis

5.1. Unified kinematics modeling

Considering the difference between the two phases of the rTPS limb, it can be found that the key part is the rotation about the radial axis which can be represented by angle $\beta$ as in Fig. 6. A limb coordinate system $\Sigma_0$ is located at the rT joint center with $x$ parallel to the bracket axis $rTba$, $y$ parallel to the bracket axis $rTy$, and $z$ perpendicular to the bracket axis.

![Fig. 5. Mobility change map.](image)

![Fig. 6. Unified limb modeling.](image)
axis collinear with the bracket axis and $y'$ axis perpendicular to the bracket surface as in Fig. 6.

It can be taken as that the $(rT)1$ limb has variable angle $\beta$ while the $(rT)2$ limb has a fixed angle $\beta = 0$. Geometrically, spherical joint center $A$ is constrained on plane $\Sigma_1$ passing through the spherical joint center and perpendicular to the bracket axis for a giving angle $\beta$ in the $(rT)1$ limb and it is constrained on plane $\Sigma_0$ passing through the $(rT)2$ joint center and perpendicular to the bracket axis in the $(rT)2$ limb. Based on Fig. 6, the spherical joint center $A$ in the limb coordinate system $\{o'x'y'z'\}$ can be given as

$$\mathbf{a} = l(-\sin \beta, \cos \beta \cos z, \cos \beta \sin z)$$

$$\beta = 0 \quad (rT)2PS$$

Based on the actuation analysis in Section 4, the rotation about the radial axis of the $(rT)1$ limb can be used as actuation input. Thus, the $(rT)2PS$ limb can be taken as a special configuration of the $(rT)2PS$ limb by locking the actuation at $\beta = 0$. This gives an important method to unify the geometric and kinematics modeling of the 3rTPS metamorphic parallel mechanism by covering all its reconfigurable topologies with mobility change. The following is to use the prismatic joint in the $(rT)2PS$ limb as actuation input and the rotation about the radial axis is added for the second actuation when the limb changes to phase $(rT)2PS$.

Based on the above analysis and the coordinate systems of the $3rTPS$ metamorphic parallel mechanism in Fig. 7, the geometric constraint of the mechanism can be given as

$$\begin{cases}
\mathbf{a}_1 = \mathbf{b}_1 + R(z,-\pi/2)R(x,-\pi/2)Rz(-\sin \beta_1, \cos \beta_1 \cos z_1, \cos \beta_1 \sin z_1) \\
\mathbf{a}_2 = \mathbf{b}_2 + R(z,-\pi/2)Rz(-\sin \beta_2, \cos \beta_2 \cos z_2, \cos \beta_2 \sin z_2) \\
\mathbf{a}_3 = \mathbf{b}_3 + R(z,-\pi/2)Rz(-\sin \beta_3, \cos \beta_3 \cos z_3, \cos \beta_3 \sin z_3)
\end{cases}$$

$$\begin{bmatrix}
\beta_1 = (rT)1PS \\
\beta_1 = 0 \quad (rT)2PS
\end{bmatrix}$$

where $\mathbf{l}_i$ is the limb length, $R(k, g)$ represents a rotation about axis $k$ with angle $g$ and is used to translate the vector of the spherical joint center in the limb coordinate systems to the global coordinate system in Fig. 7.

5.2. Inverse and forward kinematics analysis

5.2.1. Inverse kinematics analysis

The inverse displacement analysis of the $3rTPS$ metamorphic parallel mechanism is to obtain the actuation parameters (limb length $l_i$, radial-axis rotation angle $\beta_i$) based on the given platform position and orientation. Attach a local coordinate system $o'x'y'z'$ at the centroid of the platform with $x'$-axis passing through spherical joint center $A_1$ and axis $y'$ parallel to line $A_2A_3$ as in Fig. 8.

When giving the platform position $\mathbf{p} (p_x, p_y, p_z)$ and orientation $\mathbf{R}$ described in the global coordinate system in Fig. 7, the position of the spherical joint centers can be obtained as

$$\begin{bmatrix}
\mathbf{a}_1 = \mathbf{p} + R(r_\theta,0,0) \\
\mathbf{a}_2 = \mathbf{p} + R(-r_\theta/2,\sqrt{3}r_\theta/2,0) \\
\mathbf{a}_3 = \mathbf{p} + R(-r_\theta/2,-\sqrt{3}r_\theta/2,0)
\end{bmatrix}$$

where the spherical joint centers on the platform are symmetrically arranged on a circle with radius $r_\theta$.

It should be mentioned that, the platform position should be given based on the mechanism topology as analyzed in Section 3. The basic rule is that when limb 1 is in phase $(rT)2PS$, position element $p_x$ is constrained by the spherical joint center vector $\mathbf{a}_1$ with its element on $y$-axis $a_{1y} = 0$. This is same for limb 2 with phase $(rT)2PS$ and $p_y$ is constrained by $a_{2y} = 0$. In limb 3 with phase $(rT)2PS$, $p_z$ is constrained based on $a_{3z} = 0$. The relation can be concluded as

$$\begin{bmatrix}
p_y = -r_\theta r_\alpha a_{1y} \quad \text{limb 1 in (rT)2PS} \\
p_z = r_\theta r_\alpha a_{1y}/2 - \sqrt{3}r_\theta^{3/2} a_{1z}/2 \quad \text{limb 2 in (rT)2PS} \\
p_z = r_\theta r_\alpha a_{1y}/2 + \sqrt{3}r_\theta^{3/2} a_{1z}/2 \quad \text{limb 3 in (rT)2PS}
\end{bmatrix}$$

where $n_j$ is the element with the $j$th row and $h$th column in the rotation matrix $\mathbf{R}$.

By correspondingly equaling (12) to (11), the inverse displacement analysis can be solved

$$\begin{bmatrix}
\mathbf{l}_1 = \sqrt{\mathbf{c}_1 \cdot \mathbf{c}_1} \\
\mathbf{l}_2 = \sqrt{\mathbf{c}_2 \cdot \mathbf{c}_2} \\
\mathbf{l}_3 = \sqrt{\mathbf{c}_3 \cdot \mathbf{c}_3}
\end{bmatrix}$$

$$\begin{bmatrix}
\beta_1 = -\sin^{-1}(c_{1x}) \quad (rT)1PS \\
\beta_1 = 0 \quad (rT)2PS
\end{bmatrix}$$

where

$$\begin{bmatrix}
\mathbf{c}_1 = R(x, \pi/2)R(z, \pi/2)(\mathbf{a}_1 - \mathbf{b}_1) \\
\mathbf{c}_2 = R(z, \pi/2)R(x, \pi/2)(\mathbf{a}_2 - \mathbf{b}_2) \\
\mathbf{c}_3 = R(z, \pi/2)(\mathbf{a}_3 - \mathbf{b}_3)
\end{bmatrix}$$

$c_{1x}$ is the element of vector $\mathbf{c}_1$ on the $x$-axis, $\mathbf{a}_i$ is expressed in (12) and $\mathbf{b}_1$ is from (4).

5.2.2. Forward kinematics

On the contrary to the inverse displacement analysis, the forward one is to solve the platform position $\mathbf{p} (p_x, p_y, p_z)$ and orientation $\mathbf{R}$ when giving the corresponding actuation parameters $(l_i, \beta_i)$ for each topology. Based on this and Fig. 8, the geometric constraint of the platform triangle can be described as

$$\begin{bmatrix}
d_{12} = (\mathbf{a}_1 - \mathbf{a}_2)^2 \\
d_{13} = (\mathbf{a}_1 - \mathbf{a}_3)^2 \\
d_{23} = (\mathbf{a}_2 - \mathbf{a}_3)^2
\end{bmatrix}$$

$$\begin{bmatrix}
\beta_i = (rT)1PS \\
\beta_i = 0 \quad (rT)2PS
\end{bmatrix}$$

Substituting (11) into (15) and replacing $\cos z_i = (1 - t_{1i}^2)(1 + t_{1i}^2), \ldots$
be solved by substituting each solution of equation (16). Based on this, sixteen pair of solutions of the forward kinematics examples in Table 1, indicating the closure validity of the inverse and forward kinematics analysis. This can be also seen that the number of input parameters increases with the topology in Table 1. In Fig. 9, the blue line is the base and the black lines represent the limbs, the platform coordinate system is shown.

\[ \sin{\delta_j} = 2t_j/(1 + t_j^2), \]

where \( f_j(\bullet) \) is a linear function of the power product in the bracket with coefficients depending on known parameters only, \( t_j \) is \( \tan(\delta_j)/2 \).

By using Sylvester’s dialytic elimination method [38,39] for the first two equations in (16), there is

\[ f_1(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8) = 0 \]

where \( f_2(\bullet) \) is a linear function of the power product in the bracket with coefficients depending on known parameters only. Then, following the same method for (17) and the third equation in (16), a polynomial with only unknown \( t_3 \) can be obtained as

\[ \sum_{i=0}^{16} h_i t_3^i = 0 \]

where coefficient \( h_i \) are real constants depending on input data only.

This shows that a univariate equation in \( t_3 \) of degree 16 is obtained.

Solving (18), 16 solutions for \( t_3 \) can be obtained. Then, \( t_3 \) can be solved by substituting each solution of \( t_3 \) back to the third equation in (16) and selecting the roots satisfying (17). Following this, \( t_3 \) can be solved by substituting each pair of solutions of \( t_3 \) and \( t_1 \) into the first equation in (16) with proof of the second equation in (16). Based on this, sixteen pair of solutions of \( t_3 \), \( t_1 \), \( t_2 \), \( t_3 \) are obtained and the spherical joint center \( \mathbf{A_i} \) can be calculated by substituting \( z_i = 2 \arctan(t_j) \) into (11). Then, the platform position and orientation can be determined using the three spherical joint centers with Fig. 8 as

\[
\mathbf{Z'} = (\mathbf{a_2} - \mathbf{a_1}) \times (\mathbf{a_3} - \mathbf{a_1})/|(|(\mathbf{a_2} - \mathbf{a_1}) \times (\mathbf{a_3} - \mathbf{a_1})|)
\]

\[
\mathbf{Y'} = (\mathbf{a_2} - \mathbf{a_1})/(\sqrt{3} r_0)
\]

\[
\mathbf{X'} = \mathbf{Y'} \times \mathbf{Z'}
\]

\[
\mathbf{R} = (\mathbf{X'}, \mathbf{Y'}, \mathbf{Z'})
\]

\[
\mathbf{p} = \mathbf{a_1} - r_0 \mathbf{X'}
\]

The real roots correspond to assembly configurations of the 3rTPS parallel mechanism.

5.3. Numerical examples of the kinematics analysis

5.3.1. Inverse kinematics examples

Based on the kinematics analysis in Section 5.2, inverse kinematics examples for the four topologies of the 3rTPS metamorphic parallel mechanism can be given as in Table 1 to demonstrate the unified kinematics modeling. The structure parameters are set as: \( r_0 = 10, n_0 = 2\sqrt{3} \). In the examples in Table 1, the four topologies correspond to the four in Figs. 2–4, which means that the 2(3rT)PS–1(3rT)PS is obtained by altering the phase of limb 1 from the topology 3(3rT)PS. Then, the 1(3rT)PS–2(3rT)PS is obtained by further altering the phase of limb 2. By giving the orientation and position parameters of the platform and using (14), the inverse kinematics can be solved with results in Table 1. The input rotation matrix is given as

\[
\mathbf{R}_0 = \begin{bmatrix}
0.0861 & 0.5560 & 0.8267 \\
-0.7223 & -0.5366 & 0.4361 \\
0.6862 & -0.6347 & 0.3554
\end{bmatrix}
\]

which is used for all the topologies in Table 1.

From Table 1, \( \mathbf{P}_0 \) is the parasitic position of the 3(3rT)PS parallel mechanism when giving the orientation \( \mathbf{R}_0 \). It can be seen that the number of input parameters increases with the mobility increasing in the topology change. Following this, the number of output parameters also increases with specific rotation angles (\( \beta_i \)) about the radial axes of the (rT)PS limbs instead of value zero in the (rT)PS phase.

5.3.2. Forward kinematics examples

In order to check the validity of the forward kinematics analysis, the same structure parameters and the output parameters in Table 1 in the inverse kinematics examples are used as the input for the forward kinematics examples. Following the procedures in Section 5.2.2, the forward kinematics can be solved. As for each topology, there are 16 solutions for the forward kinematics analysis, only the solutions for topology 3(3rT)PS are listed in Table 2 as an example. Real solutions of all the topologies are listed in Table 3 with corresponding mechanism assemblies demonstrated in Fig. 9.

In Table 3, it can be found that one of the real solutions of the forward kinematics corresponds to the input one of the inverse kinematics examples in Table 1, indicating the closure validity of the inverse and forward kinematics analysis. This can be also seen in Fig. 9 that the orientation of the platform (green triangle) is the same in Fig. 9 (b), (d), (h) and (l), corresponding to the rotation matrix \( \mathbf{R}_0 \) in (20) which is the input of the inverse kinematics analysis in Table 1. In Fig. 9, the blue line is the base and the black lines represent the limbs, the platform coordinate system is shown.
with blue arrow in x direction, black arrow in y direction and the red arrow in the z direction perpendicular to the platform plane.

6. Conclusions

This paper introduced a new metamorphic parallel mechanism named 3rTPS consisting of three reconfigurable rTPS limbs in perpendicular base planes. The rTPS limb had two working phases resulted from the configuration change of the reconfigurable Hooke (rT) joint. Screw theory was used to analyze mobility of the rTPS limb in its two phases, which showed that in one phase the rTPS limb has mobility 6 and in the other it had one DOF less, providing a constraint force to the platform. Based on this, the platform constraint screw system demonstrated the topology

Table 3
Real solutions of forward kinematics.

<table>
<thead>
<tr>
<th>Topology</th>
<th>Real solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>3rTPS</td>
<td>( R_{s1} = \begin{pmatrix} -0.3058 &amp; 0.5142 &amp; 0.8013 \ -0.2824 &amp; -0.8527 &amp; 0.4395 \ 0.9092 &amp; 0.0918 &amp; 0.4059 \end{pmatrix} ); ( R_{s2} \rightarrow R_{s0} ); ( p_{s1} = (1.0131, 0.9782, 1.8505) ); ( p_{s2} = (1.8171, 2.5022, 3.0925) ).</td>
</tr>
<tr>
<td>2rTPS–1rTPS</td>
<td>( R_{s1} = \begin{pmatrix} -0.2888 &amp; 0.4111 &amp; 0.8646 \ -0.1683 &amp; -0.9108 &amp; 0.3768 \ 0.9424 &amp; 0.0366 &amp; 0.3232 \end{pmatrix} ); ( R_{s2} \rightarrow R_{s0} ); ( p_{s1} = (0.733, 1.983, 1.7424) ); ( p_{s2} = (1.8171, 3.9022, 3.0925) ).</td>
</tr>
<tr>
<td>1rTPS–2rTPS</td>
<td>( R_{s1} = \begin{pmatrix} -0.1775 &amp; 0.4282 &amp; 0.8861 \ 0.0039 &amp; -0.9001 &amp; 0.4357 \ 0.9841 &amp; 0.0808 &amp; 0.1591 \end{pmatrix} ); ( R_{s2} \rightarrow R_{s0} ); ( p_{s1} = (0.5171, 0.5462, 4.292) ); ( p_{s2} = (0.733, 1.983, 1.7424) ); ( p_{s3} = (1.5367, 1.5516, 3.23) ); ( p_{s4} = (1.8171, 3.9022, 4.292) ).</td>
</tr>
<tr>
<td>3rTPS</td>
<td>( R_{s1} = \begin{pmatrix} 0.1286 &amp; 0.4409 &amp; 0.8882 \ -0.0452 &amp; -0.8921 &amp; 0.4495 \ 0.9906 &amp; 0.0979 &amp; 0.0948 \end{pmatrix} ); ( R_{s2} \rightarrow R_{s0} ); ( p_{s1} = (0.2457, 1.556, 3.2098) ); ( p_{s2} = (0.391, 1.683, 3.402) ); ( p_{s3} = (0.546, 2.079, 3.786) ); ( p_{s4} = (0.5171, 3.9022, 4.292) ).</td>
</tr>
</tbody>
</table>
change of the 3rTPS metamorphic parallel mechanism by altering the phases of the rTPS limbs, which illustrated that the mechanism had four topologies with ability of mobility change among 3R (three rotations), 3R1T (three rotations one translation), 3R2T and mobility 6. Following this, the actuation constraint screws showed that the rotations about the bracket axis and the radial axis, the translation of the prismatic joint can be chosen as the actuation input for the variety topologies. This gave the possibility of a unified kinematics modeling of the rTPS limb by choosing the rotation about the radial axis as the actuation and taking one phase as a special case of the other. Based on this, a unified kinematics modeling of the 3rTPS metamorphic parallel mechanism was obtained to cover all its four topologies. Analytical solutions of inverse and forward kinematics analysis were obtained. Numerical examples were used to demonstrate the analysis with mechanism configurations illustrated corresponding to the real solutions, which proved the validity of these theoretical results.

References

7. Saglia JA, Dai JS, Caldwell DG. Geometry and kinematic analysis of a redundantly actuated parallel mechanism that eliminates singularities and improves dexterity. ASME Journal of Mechanical Design 2008;130 12450-1-5.