1 Introduction

Parallel mechanisms that have pure rotation motion are called spherical parallel mechanisms (SPMs) and are a class of mechanisms that have attracted much interest in the mechanisms research community due to their usefulness in real applications and ability to decouple rotations. Novel applications of SPMs include camera orienting devices [1], robotic wrists [2–4], shoulders [5] and hips [6], robotic palm [7,8], robotic surgery [9], and human joint rehabilitation [10].

Generally there are two types of SPMs [11]: (a) over-constrained (represented by $3RR$ [1]) and (b) nonover-constrained (represented by $3SPs$-$1RS$ [12–14] and $3UPU$ [15–18]). The former mechanisms are normally actuated by revolute joints while the latter mechanisms are actuated by prismatic joints and are called linear-actuated spherical parallel mechanisms (LASPMs). These two types of mechanisms have different kinematics constraints. However, when the over-constrained mechanisms are actuated by different joints, like the middle $R$ joint in a $3RR$ mechanism, they can be considered equivalent to LASPMs. LASPMs with a symmetrical limb arrangement and symmetrical base and platform shapes are called LASSPMs and are the focus of this paper. The main motivation is that symmetrical spherical parallel manipulators are commonly used ones since they provide symmetrical kinematics and dynamics properties. The work in this paper provides a method to easily analyze LASSPMs and will be a base for further optimization design of those SPMs with clearer criteria covering kinematics, singularity, and workspace. Furthermore, the work also provides a simpler kinematics model for the existing LASSPMs in their dynamics and control. Rotation motion results in nonlinear geometric constraint equations, which is reflected in the forward kinematic equations of SPMs. Early research has shown that the forward kinematic equations of a general SPM have eight solutions [12,19]. Echelon form, including two polynomials of order eight and one, was obtained in Ref. [12] to solve for the eight FK solutions. A special coordinate system was used in Ref. [19] to simplify the kinematics equations using Euler angles. Actuation joint angles were used as unknowns in solving the FK in Ref. [20] and a closed-form solution for a specific structure was obtained in Refs. [21,22]. A robust forward kinematic analysis was proposed in Ref. [23], by representing the SPMs using two spherical four-bar linkage mechanisms, and the solutions were obtained semigraphically. Recently, Ref. [24] demonstrated that there are four trivial and four nontrivial FK solutions of the Agile eye SPM. The four trivial ones are singular configurations while the four nontrivial ones correspond to four assemble modes and can be uniquely determined for a given set of actuation inputs. A simple kinematics equation was proposed in Ref. [25] to calculate the unique solution of the Agile eye. This work was then extended to a LASSPM with orthogonal base and platform [26], where the forward kinematic equations were analytically solved. The eight solutions in singularity-free
zones were used to find the current FK solution. However, the method cannot be applied to other symmetrical SPMs like those in this paper.

This paper introduces the Cayley formula [27] to represent the rotation matrix, when deriving the kinematic equations of LASSPMs, resulting in a set of simple, analytical forward kinematics equations in terms of the three Rodriguez–Hamilton parameters. Cayley formula provides three independent unknowns to describe 3D rotation motion and showed advantage in simplifying FK of Stewart–Gough platforms [28–30] and a spherical parallel wrist mechanism [14]. The Euler parameters, the homogenized version of the Rodriguez–Hamilton parameters, were also used extensively in parallel mechanism kinematics [31–33]. However, they both were not applied in singularity and workspace analysis which is a novel point in this paper which clearly illustrates the singularity loci and singularity-free workspace.

An important topic of SPMs is the analysis of type 1 and type 2 singularities [34]. Type 1 singularities represent the workspace boundaries while type 2 singularities are generally inside the workspace and are commonly studied [35]. Singularity loci of SPMs are not difficult to derive since there are only three unknowns, e.g., ZYZ [36] and XZY [37] Euler angles. However, it is difficult to interpret and represent singularity loci explicitly with clear physical meaning. Two special cases were illustrated in Ref. [38], in both joint space and Cartesian space. By using the tilt-and-torsion (T&T) angles [39], analytical expressions for type 2 singularity loci of the 3RR and 3SPS-1S mechanisms were obtained. Following that, the authors [40] extended the study to type 1 singularity loci, by explicitly representing the workspace boundaries. Much work has been done on rotation workspace analysis and representation of parallel mechanisms, based on exponential coordinates [41], Euler angles [37], 2D projections [42], and geometric parameterizations [43]. However, most work used numerical methods in the solutions and [40] appears to be the only study to derive explicit workspace boundaries. Constraint singularities are also common for SPMs [44] but they are covered in the type 2 singularity loci in this paper and not pointed out in the analysis as this is not the focus.

The kinematic model proposed in this paper gives a new way to represent the singularity loci with clear physical meaning, and also shows a new approach to obtain analytical expressions for workspace boundaries. In addition to this, the unified model has the advantage of presenting the singularity-free workspace, by combining the manipulator workspace and singularity loci in the same coordinate system. The paper is arranged as follows. Section 2 introduces the LASSPMs and their forward kinematic analysis based on geometric constraints. Singularity loci are derived in Sec. 3 and the FK solution in the singularity-free zones are obtained in Sec. 4. Section 5 shows the analytic expressions of the workspace boundaries with example singularity-free workspace representations. Conclusions are made in Sec. 6.

2 LASSPMs and Their FK Analysis

2.1 LASSPMs and Their Geometric Constraints. Two well-known linear actuated SPMs are the 3SPS-1S and 3UPU mechanisms, Figs. 1(a) and 1(b). In the 3SPS-1S parallel mechanism, the platform is constrained by the S limb to have pure rotation around this S joint center with actuation from the prismatic joints in the three SPS limbs. Each prismatic joint controls the distance between the platform spherical joint and the base spherical joint. Similarly, the two rotation axes of the two universal joints in each limb in the 3UPU wrist mechanism intersect at one point which virtually constrains the platform to rotate purely around this intersecting point, Fig. 1(b). The prismatic joints in the three limbs are used to control this rotation motion by changing the distance between the two universal joints in each limb. The 3RRR SPM is not linear actuated and the first R joint on the base in each limb is normally the actuation joint, like the Agile eye [1]. However, when actuating the middle R joint in the limb, the 3RRR can be considered as a linear actuated SPM, Fig. 1(c), as it controls the distance between the platform and base R joint centers. Based on this rule, all SPMs which control their rotation motion by equivalently controlling the distance between the platform and base joint centers can be considered as LASSPMs. In this paper, we consider LASSPMs. Here symmetry means, symmetrically arranged limbs and symmetrical (similar) platform and base shapes. In the following, only three-leg LASSPMs are considered, as extra limbs are redundant.

A representative kinematics model of a general LASSPM is shown in Fig. 1(d), where the three limbs support the platform symmetrically around a reference circle of radius r, and connect to the base around a circle of radius r. The rotation center is at point O and the three limbs are numbered limb 1, limb 2 and limb 3. Let B, in the base denote the center of the base joint and , denote the center of the platform joint of the ith (i = 1, 2, 3) limb. Set a base coordinate system O-xyz at the center O with z-axis perpendicular to the base plane formed by B1B2B3, and x-axis perpendicular to the line OB1, Fig. 1(d). Attach a platform coordinate frame O-x'y'z' at the rotation center O with z' axis perpendicular to the platform and x-axis perpendicular to OA1. When the platform is at the initial configuration, the platform coordinate frame is coincident with the base coordinate frame. Lines O'A1 and OB form platform angle x and base angle x0, respectively, with the z'-axis and z-axis. These platform and base angles represent the shapes of the platform and base. When the platform and base angles are equal, x = x0, the platform and the base have similar shapes, with symmetry, and these LASSPMs are studied in this paper. Angle x = x0 = x is used in the rest of this study. When x = sin⁻¹(√2/3), a special LASSPM with orthogonal base and platform is obtained [26].

Let q denote the constant position vector of platform joint center A1 in the platform coordinate frame O-x'y'z' and b1 be the constant vector of base joint center B1 expressed in the base coordinate frame O-xyz. Then the limb distance constraints can be described as
where $d_i$ is the length of limb $i$, $\mathbf{R}$ is the $3 \times 3$ rotation matrix.

Expanding Eq. (1) gives

$$2(\mathbf{R} \cdot \mathbf{a}_i)^T \cdot \mathbf{b}_i = a_i^2 + b_i^2 - d_i^2 = 2a_ib_i\cos(\phi_i) \quad (i = 1, 2, 3)$$

which can be described as

$$(\mathbf{R} \cdot \mathbf{v}_i)^T \cdot \mathbf{u}_i = \cos(\phi_i) \quad (i = 1, 2, 3)$$

where $\mathbf{v}_i = a_i/|a_i|$ and $\mathbf{u}_i = b_i/|b_i|$ denote the unit vectors along $OA_i$ and $OB_i$ in the platform coordinate system and the base coordinate system respectively, and $\phi_i$ is the angle between axes $\mathbf{v}_i$ and $\mathbf{u}_i$, Fig. 1(a).

Equation (3) gives the general geometric constraint of LASSPMs and states that the key constraint is the angle between the platform joint vector ($\mathbf{v}_i$) and base joint vector ($\mathbf{u}_i$) of each limb. Equation (3) will be used in the kinematics and assembly mode analysis. It is noted that given $\mathbf{R}$ the inverse kinematic solution can be obtained directly from Eq. (1) to give the input limb lengths $d_i$. The FK analysis in general is more complex and Sec. 2.2 shows a simple solution.

### 2.2 FK

When a rigid body rotates by an angle $\theta$ about an axis $k(x_k, y_k, z_k)$ in a three-dimensional coordinate system, the rotation matrix $\mathbf{R}$ is given by

$$\mathbf{R}_b(\theta) = \begin{bmatrix}
-k_1^2\theta + c_\theta & k_1k_3\theta - k_2s_\theta & k_1k_2\theta + k_3s_\theta \\
-k_1k_3\theta + k_2s_\theta & k_1^2\theta + c_\theta & -k_1k_2\theta + k_3s_\theta \\
k_1k_2\theta - k_3s_\theta & k_1k_3\theta + k_2s_\theta & k_1^2\theta + c_\theta
\end{bmatrix}$$

where $c_\theta = \cos \theta$, $s_\theta = \sin \theta$, and $\theta = 1 - \cos \theta$

Let

$$c_1 = k_1 \tan(\theta/2)$$
$$c_2 = k_2 \tan(\theta/2)$$
$$c_3 = k_3 \tan(\theta/2)$$

$$\cos (\theta) = (1 - \tan^2(\theta/2))/(1 + \tan^2(\theta/2))$$
$$\sin(\theta) = 2 \tan(\theta/2)/(1 + \tan^2(\theta/2))$$

where $c_1$, $c_2$, and $c_3$ are the Rodriguez–Hamilton parameters. The rotation angle $\theta$ is between $(-\pi, \pi)$; thus $\tan(\theta/2)$ is less than infinity and Eq. (5) does not have any singularities.

Substituting Eq. (5) into Eq. (4) yields

$$\mathbf{R} = \Delta^{-1} \begin{bmatrix}
1 + c_1^2 - c_2^2 - c_3^2 & 2(c_2c_3 - c_1) & 2(c_1c_3 + c_2) \\
2(c_1c_2 - c_3) & 1 - c_1^2 + c_2^2 - c_3^2 & 2(c_2c_3 + c_1) \\
2(c_3c_2 + c_1) & 2(c_2c_3 - c_1) & 1 - c_1^2 - c_2^2 + c_3^2
\end{bmatrix}$$

$$\Delta = 1 + c_1^2 + c_2^2 + c_3^2$$

where $\Delta$ is the Cayley formula [27].

Substituting Eq. (6) into Eq. (3), simplifying and taking the numerators, gives

$$f_i(1, c_1, c_2, c_1c_2, c_1^2, c_2^2) = 0 \quad (i = 1, 2, 3)$$

where $f_i(\cdot)$ is a function of the unknown power products in the bracket, with real constant coefficients depending on the input and mechanism dimension parameters only. This will be used in the following sections in this paper without further introduction.

The three equations in Eq. (7) contain the square of $c_3$ and this can be eliminated giving two equations in the two unknowns $c_1$ and $c_2$

$$f_i(1, c_1, c_2, c_1c_2, c_1^2, c_2^2) = 0 \quad (i = 4, 5)$$

Then by using Sylvester resultant to eliminating $c_2$, a single equation in the single unknown $c_1$ is obtained

$$f_6(1, c_1, c_1^2, c_1^3) = 0$$

Solving Eq. (9), four solutions for $c_1$ can be obtained. Then substituting $c_1$ into equations in Eq. (8) and using the Euclidean method, four corresponding solutions for $c_2$ can be obtained. Based on each of the four sets of solutions of $c_1$, $c_2$ and any equation in Eq. (7), two conjugate solutions of $c_3$ can be obtained. Hence there are eight sets of solutions for the forward kinematic equations. The LAQSPM [26] with orthogonal base and platform can also be solved directly using the above method.

### 3 Singularity Loci

#### 3.1 Jacobian Matrix and 3D Singularity Loci

Taking the derivative of Eq. (1), there is

$$\begin{bmatrix}
\mathbf{b}_1 \times (\mathbf{R} \cdot \mathbf{a}_1) \\
\mathbf{b}_2 \times (\mathbf{R} \cdot \mathbf{a}_2) \\
\mathbf{b}_3 \times (\mathbf{R} \cdot \mathbf{a}_3)
\end{bmatrix} \omega = \begin{bmatrix}
\mathbf{J}_1 \\
\mathbf{J}_2 \\
\mathbf{J}_3
\end{bmatrix} \omega = \mathbf{J}_0 = \begin{bmatrix}
d_1 \\
d_2 \\
d_3
\end{bmatrix} \begin{bmatrix}
d_1 \\
d_2 \\
d_3
\end{bmatrix}$$

where $\omega$ is the platform orientation velocity, $d_i$ is the input velocity of limb $i$, and $\mathbf{J}_i = (\mathbf{b}_i \times (\mathbf{R} \cdot \mathbf{a}_i))^T$ is the three row vectors of the Jacobian matrix $\mathbf{J}$ of the LASSPM. Hence type 2 singularities result when the determinant of $\mathbf{J}$ equals to zero. Based on the rotation matrix $\mathbf{R}$ in Eq. (6) and the symmetrical structure, the determinant of $\mathbf{J}$ is given by

$$\det(\mathbf{J}) = f_5(c_1^2, c_1^3, c_1^4, c_2^2, 1) \times c_1/\Delta^2$$

It is noted that Eq. (11) is a polynomial of order 3 in $c_1$ and order 2 in $c_2$ with a factor $c_3$. When $c_3 = 0$, the determinant equals to zero, indicating that the mechanism is always in a singular configuration if there is no rotation about the $z$-axis (represented by rotation element $c_3$). This is due to the fact that the three row vectors in the Jacobian matrix $\mathbf{J}$ are always in the same plane when there is no $z$-axis rotation, resulting in one of them being redundant and making the determinant zero. This can be avoided in practical applications by making the initial assembly of the mechanism with a finite rotation about the $z$-axis.

From Eq. (11), the determinant of the Jacobian matrix is a cubic polynomial of the three rotation elements ($c_1$, $c_2$, and $c_3$) with coefficients consisting of structure parameters. For a given structure of a LASSPM, the rotation elements ($c_1$, $c_2$, and $c_3$) can be used to represent the singularity points of the platform. By equaling Eq. (11) to zero, all the singular points can be found and these are illustrated in Fig. 2.

From Eq. (11), it is seen that $c_1$ is a factor of the determinant, and hence the plane $c_3 = 0$ is a part of the singularity loci, and the other parts of the singularity loci depend only on $c_1$ and $c_2$. This can be seen from an example LASSPM with $x = \pi/4$, Fig. 2(a), where the singularity loci consists of the $c_3 = 0$ plane, and three other scattered parts and a central part on surfaces perpendicular to the $c_3 = 0$ plane. The singularity loci surface has clear symmetry on the $c_1c_2$ plane as a result of the symmetrical limb arrangement and is also symmetrical with respect to the $c_2 = 0$ plane, due to the elements of $c_2$ in the Jacobian matrix determinant being quadratic.

When $x$ varies, the singularity loci also change accordingly, as illustrated in Fig. 2(b) with $x = \sin^{-1}(\sqrt{2}/3) \approx 0.95$ rad with orthogonal base and platform. The special case in Fig. 2(b) shows that when the mechanism has orthogonal base and platform, the
singularity loci become three intersecting planes. This can be explained from the Jacobian matrix determinant (11) which can be expressed as

$$\begin{align*}
|J| &= 3\sqrt{3}a_p^3 \left( -2 + 3c_1^2 + 3c_2^2 - 3\sqrt{2}c_1c_2 \right) c_3/\Delta^2 \\
&= 12\sqrt{3}a_p^3 J_{12} J_{32}/\Delta^2
\end{align*}$$

(12)

where \( J_1 = (c_1 - 1)/\sqrt{2}, \) \( J_2 = (-c_1/2 + \sqrt{3}/2 - 1/\sqrt{2}), \) \( J_3 = (-c_1/2 - \sqrt{3}/2 - 1/\sqrt{2}), \) and \( J_4 = c_2. \)

This shows that the singularity surface consists of four planes represented by \( J_i \) (\( i = 1, 2, 3, 4 \)) as in Fig. 2(a). In addition to the \( J_4 = c_3 = 0 \) plane, the other three are perpendicular to the \( J_4 = c_3 = 0 \) plane and symmetrically distributed with intersection angle \( 2\pi/3 \), while intersecting at three parallel lines.

When further increasing \( z \) to \( z = \pi/3 \), Fig. 2(c), the singularity loci becomes similar to \( z = \pi/4 \) case. This evolution is demonstrated in Fig. 2(d), from the top view of the singularity loci. It can be seen that when \( z \) is closer to \( \sin^{-1}(\sqrt{2}/3) \approx 0.95 \) rad, that is when the mechanism is closer to having orthogonal base and platform, the three scattered parts of the singularity loci is closer to the central part and the central part becomes larger. The four parts connect to one another when \( z = \sin^{-1}(\sqrt{2}/3) \).

3.2 Limb Actuation Singularity. Limb actuation singularity [26] defines configurations where the limb cannot be actuated even when the other limb actuation joints are released. In the LASSPMs, it happens when a limb passes through the rotation center \( O \). The following analysis shows a new method to calculate the limb actuation singularity loci.

Based on the Jacobian matrix in Eq. (10), limb actuation singularities can be found by making the row vector \( J_i = (J_{1i}, J_{2i}, J_{3i}) = 0 \). In general, the components \( J_{ij} \) are quadratic polynomials of the rotation elements \( (c_1, c_2, c_3) \) and \( J_i = (J_{1i}, J_{2i}, J_{3i}) = 0 \) gives three curved surfaces intersecting at two lines as shown in Fig. 3.

![Fig. 2 Singularity Loci of LASSPMs. (a) \( z = \pi/4 \), (b) \( z = \sin^{-1}(\sqrt{2}/3) \approx 0.95 \) with orthogonal base and platform, (c) \( z = \pi/3 \), and (d) variable \( z \) and the singularity loci evolution.](http://mechanismsrobotics.asmedigitalcollection.asme.org/)![Fig. 3 Limb actuation singularity loci. (a) Leg 1 \( (J_2 = 0) \) and the mechanism singularity configurations, (b) leg 2 \( (J_2 = 0) \), and (c) leg 3 \( (J_3 = 0) \).](http://mechanismsrobotics.asmedigitalcollection.asme.org/)
Fig. 4 Limb actuation singularity loci in the mechanism singularity loci. (a) $x = \pi/4$ and (b) $x = \sin^{-1}(\sqrt{2}/3)$ with orthogonal base and platform.

Fig. 5(a), represents an anticlockwise rotation of angle $\theta$ about the line $OC$. When $\theta = \pi$, point $C$ goes to infinity. Similarly, the point $C'$ in the opposite direction of line $OC$ will go to the infinity, when it rotates anticlockwise by angle $\theta = \pi$, about the line $OC'$. Physically, the two rotations are about the same line but in opposite directions, with rotation angle $\theta = \pi$, indicating that they reach the same configuration. Thus, point $C$ coincides with $C'$ at infinity, indicating that the two ends of any line in the $(c_1, c_2, c_3)$ coordinate system connect to each other. This will be used in dividing the singularity free zones for LASSPMs with orthogonal base and platform.

As shown in Fig. 2(b), the 3D space of $(c_1, c_2, c_3)$ is divided into 14 singularity-free zones by the four singularity planes in Eq. (12). The parts with positive $J_i = c_i > 0$ are numbered from zone $z1$ to zone $z7$, Fig. 5(b). Based on the above connectivity analysis, the zone with negative $c_1$ under zone $z1$ is connected to zone $z4$ with the lines in these zones go to infinity and the two are considered as one zone, $z4$. Similarly, the part with negative $c_3$ under zone $z4$ is connected to zone $z1$. This is the same for the other zones, e.g., the part under zone $z2$ belongs to zone $z5$. However, zone $z7$ is different, as zone $z7$ is connected to its negative part with only one line $(c_1 = 0, c_2 = 0)$. Thus, the part under zone $z7$ is defined as zone $z8$. Those eight zones can be described by the signs of the four equations $J_i$ ($i = 1, 2, 3, 4$) as in Table 1.

It is shown that the eight FK solutions represent eight assembly modes and are distributed among eight singularity-free zones [26]. Following this, it is found that the eight FK solutions are distributed among these eight zones $z1$ to $z8$, as illustrated by an example in Fig. 5(c), and the eight solutions are listed in Table 2. In Fig. 5(c), only the solutions with positive $c_3$ are shown as the other four are directly under them with negative $c_3$. It is seen that the solutions are distributed as follows: solution $S4$ in zone $z1$, $S3$ in $z2$, $S7$ in $z3$, and $S8$ in $z7$. Correspondingly, solution $S2$ having negative $c_1$ of $S4$ is in zone $z4$, $S1$ with negative $c_1$ of $S3$ in $z5$, $S5$ with negative $c_3$ of $S7$ in $z6$, and $S6$ with negative $c_3$ of $S8$ in $z8$. This is also shown in Table 2. Based on this, the eight zones correspond to eight assembly modes of the LASSPM with orthogonal base and platform. For a given solution, the corresponding assembly zone can be determined by checking the signs of $J_i$ in Table 1. Once the mechanism is assembled in one mode, the mechanism will keep in that mode unless it reaches a singularity configuration, and then change to another mode. Thus, the current FK solution can be uniquely found within the corresponding assembly zone, as shown in Fig. 5(b).

4.2 FK Solution Distribution for General Cases. Following the analysis in Sec. 3.3, for the general LASSPMs, the assembly zones for an example with $x = \pi/4$ is shown in Fig. 6. It is clear that zones $z1$, $z3$, and $z5$ are connected, and also zones $z2$, $z4$, and $z6$ are connected. Thus the assembly modes in these zones are not separated by the singularity surfaces and can change to each other freely. However, in general, the FK solutions are still distributed among these six zones. The solutions in zone $z7$ and $z8$ can still be uniquely determined when the mechanism works in these two assembly modes. To demonstrate this, an example is illustrated in Fig. 6. Solutions $S8$ and $S2$ are in zones $z1$ and $z4$, $S4$ and $S3$ are in $z3$ and $z6$, $S1$ and $S5$ are in $z5$ and $z2$ while $S7$ and $S6$ are in $z7$ and $z8$, respectively.

5 Analytical Singularity-Free Workspace

5.1 Workspace Representation. In Sec. 3, the rotation elements $(c_1, c_2, c_3)$ are used to illustrate the singularity loci by using a 3D coordinate system, $O-c_1c_2c_3$. This can be extended to represent the rotation workspace of LASSPMs, using the same coordinate system, with $c_1$, $c_2$, and $c_3$ in three perpendicular directions, Fig. 7(a). According to Eq. (5), a point $C(c_1, c_2, c_3) = \tan(\theta/2) \times (kx, ky, kz)$ corresponds to a platform rotation by angle $\theta$ about an axis $k(kx, ky, kz)$ in the mechanism base coordinate system, $O-c_1c_2c_3$. Thus, the workspace coordinate system, $O-c_1c_2c_3$, is parallel with the mechanism base coordinate system.

Table 1 Singularity-free zones

<table>
<thead>
<tr>
<th>Zone</th>
<th>$J_1$, $J_2$, $J_3$, $J_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z1$</td>
<td>$(+, +, +, +)$</td>
</tr>
<tr>
<td>$z2$</td>
<td>$(+, +, +, -)$</td>
</tr>
<tr>
<td>$z3$</td>
<td>$(-, +, +, +)$</td>
</tr>
<tr>
<td>$z4$</td>
<td>$(-, +, +, -)$</td>
</tr>
<tr>
<td>$z5$</td>
<td>$(+, -, +, +)$</td>
</tr>
<tr>
<td>$z6$</td>
<td>$(-, -, +, +)$</td>
</tr>
<tr>
<td>$z7$</td>
<td>$(-, -, +, -)$</td>
</tr>
<tr>
<td>$z8$</td>
<td>$(-, -, -, -)$</td>
</tr>
</tbody>
</table>

Fig. 5 Assembly zones of the LASSPM with orthogonal base and platform. (a) Connection at infinity, (b) assembly zones, and (c) FK solution distribution.
O-xyz, with coincident centers, O. This property shows that the O-c1c2c3 coordinate system has direct physical meaning in representing the LASSPM workspace. Trajectory plans can also be done in the workspace coordinate system. An arc trajectory in the O-c1c2c3 coordinate system is given by (c1 = 0.5 cos(φ), c2 = 0.5 sin(φ), c3 = 0.3) with (π/18 ≤ φ ≤ 4π/9), as shown by the blue arc in Fig. 7(a). The corresponding mechanism motion trace is shown in Fig. 7(c) where φ grows from π/18 to 4π/9.

5.2 Analytical Description of the Workspace Boundaries.

In the LASSPMs, each limb length has two limits (lower and upper) which constrain the actuation range and determines the rotation workspace of the platform. Based on Eq. (2), the limb length limits will result in lower and upper limits of angle /i between the platform and base vectors v_i and u_i in each limb. Thus, the platform rotation workspace boundaries can be expressed by the two limits /i_max/i_min, by calculating the triangle relation, using Eq. (3) as

\[(R \cdot v_i)^T \cdot u_i = \cos(\phi_{i_{\text{max/min}}}) \quad (i = 1, 2, 3)\]  \hspace{1cm} (14)

This can be expanded to give

\[f_{i\alpha}(1, c_1, c_2, c_1c_2, c_1^2, c_2^2) - r_{ab}(1 + \cos(\phi_{i_{\text{max/min}}}))c_3^2/\sin^2(\phi/2) = 0 \quad (i = 1, 2, 3)\]  \hspace{1cm} (15)

Equation (15) shows the advantage of the (c1, c2, c3) coordinate system, since only the square product of c3 appears in each of the three limb constraint equations. Hence, c3 can be easily expressed using c1 and c2 with coefficients including the angle limits. This makes it possible to have analytical expressions to describe the 3D rotation workspace using the three rotation elements. In Eq. (15), the mechanism dimension parameters (platform and base radius r_a, r_b, and angle z) cannot be zero. Then, the coefficient of c3^2 is zero only when \(\phi_{i_{\text{max/min}}} = \pm \pi\), which is a limb actuation singularity configuration. In this case, the workspace boundaries
degenerates from general surfaces to the three limb actuation singularity lines $l_i$ ($i = 1, 2, 3$) as shown in Figs. 3 and 4. This shows similar results to the degeneration analysis in Ref. [40], indicating that the workspace is no longer limited by surfaces, but also curves.

Except the above special case, $c_3$ in general can be expressed as

$$c_3 = \pm \sqrt{f_1(1, c_1, c_2, c_1 c_2, c_1^2, c_2^2) \sin^2 (x) / r z b \left(1 + \cos \left(\phi_{\text{max} / \text{min}}\right)\right)}$$

$$(i = 1, 2, 3)$$

(16)

The two solutions of $c_3$ represent the symmetrical property of the workspace with respect to the $c_3 = 0$ plane; one of these can be calculated, while the other is simply obtained using this symmetry property. Based on the workspace representation in Sec. 4.1, the 3D workspace boundaries can be represented by calculating the analytical expression (16) with input limits $\phi_{\text{max} / \text{min}}$. Since the limb arrangement in LASSPMs is symmetrical, the workspace boundaries corresponding to each of the three legs have a similar shape, but rotated $2\pi / 3$ about the $c_3$ axis. Thus, the calculation can be further simplified by calculating limb 1 ($i = 1$ in Eq. (16)) only and getting the other two solutions by rotation to obtain the whole workspace boundaries of the LASSPMs.

### 5.3 Singularity-Free Workspace in Different Assembly Zones

Following the method in Sec. 4.2, some examples are given below to demonstrate the workspace boundaries in different assembly zones. An example of a LASSPM with $x = \pi / 4$, and $\phi_{\text{min}} = 0.6, \phi_{\text{max}} = 2.1$ is shown in Fig. 8. The boundaries of limb 1 (from Eq. (16)) are shown in Fig. 8(a) where the two surfaces represent the lower and upper boundaries corresponding to the two limb input limits $\phi_{\text{max} / \text{min}}$ and the space between the two blue surfaces is the workspace. By rotating the limb one boundaries by $2\pi / 3$ about the $c_3$ axis and taking the intersection of the space between the blue surfaces, the mechanism workspace boundaries can be obtained, Fig. 8(b). The workspace is symmetrical with respect to the $c_3 = 0$ plane and the $c_1 = c_2 = 0$ line due to the symmetrical mechanism structure. By combining this workspace with the singularity loci (from Sec. 3.1), the singularity-free workspace among all the assembly zones is clearly seen, Fig. 8(c). The mechanism has a relatively smaller workspace in zones $z7$ and $z8$ than in the other zones where the workspaces are connected between the zones, based on which the mechanism can change to different zones by passing singularity surfaces.

Another example of a LASSPM with orthogonal base and platform and the same input limits as the above example is shown in Fig. 9. The workspace boundaries look different to the previous ones; the mechanism has to change the assembly zones among $z1/z4, z3/z6, z5/z2–z7/z8$. The workspace in zones $z7$ and $z8$ is also larger than that in the first example in Fig. 8.

One more example is given for a LASSPM with orthogonal base and platform, this time with a smaller input range, $\phi_{\text{min}} = 0.8$ and $\phi_{\text{max}} = 1.5$, Fig. 10. Due to the smaller input limits, the workspace between the assembly zones are separated, indicating that once the mechanism is assembled in one mode it cannot change to another one even via singularity surfaces. The mechanism only has workspaces in zones $(z1, z3, z5)$ with positive $c_3$, zones $(z2, z4, z6)$ with negative $c_3$, and zones $z7$ and $z8$. Comparing with the example in Fig. 9, workspaces in zones $z7$ and $z8$ are smaller due to the smaller input range.

### 6 Conclusions

This paper sets up a systematic method using Cayley formula and Rodriguez parameters to provide simpler kinematics solutions, clear singularity loci representation, and clear and analytic singularity-free workspace illustration of LASSPMs. The FK is solved analytically from a fourth-order polynomial in one of the three Rodriguez–Hamilton parameters as an unknown. Based on a new proposed coordinate system, 3D singularity loci are analytically derived and illustrated with clear corresponding mechanism configurations using the Rodriguez–Hamilton parameters. Based on this, a unique forward kinematic solution of the LASSPM with orthogonal base and platform is determined, giving eight solutions distributed in eight singularity-free zones. However, general LASSPMs were found not to have this property, since their singularity-free zones are connected. By expressing one of the three unknowns as a function of the other two, analytic expressions for workspace boundaries are obtained. Different examples are presented and it is found that the singularity-free workspace in different singularity-free zones could be connected when the actuation range is large and separated when the actuation range is small.

The work in this paper provides a method to easily analyze the LASSPMs and will be a base for further optimization design of those SPMs with clear criteria. Furthermore, the work also provides simpler kinematics model for the existing SPMs in their dynamics and control.

### References


