Variable Motion/Force Transmissibility of a Metamorphic Parallel Mechanism With Reconfigurable 3T and 3R Motion

This paper presents a metamorphic parallel mechanism (MPM) which can switch its motion between pure translation (3T) and pure rotation (3R). This feature stems from a reconfigurable Hooke (rT) joint of which one of the rotation axes can be altered freely. More than that, based on the reconfiguration of the rT joint, workspace of both 3T and 3R motion can be tunable, and the rotation center of the 3R motion can be controlled along a line perpendicular to the base plane. Kinematics analysis is presented based on the geometric constraints of the parallel mechanism covering both 3T and 3R motion. Following this, screw theory based motion/force transmission equations are obtained, and their characteristics are investigated and linked to the singularity analysis using Jacobian matrix. Motion/force transmission indices can be used to optimize basic design parameters of the MPM. This provides reference of this mechanism for potential applications requiring 3T and 3R motion. [DOI: 10.1115/1.4032409]

Keywords: motion/force transmissibility, reconfiguration, unified kinematics, screw theory, parallel mechanism

1 Introduction

In 3D space, a rigid body can have three translations along and three rotations about three orthogonal directions. In parallel mechanism research, this can be represented by the platform motion of the general Stewart mechanisms [1,2] which have been applied successfully in industrial assembly [3], manufacturing [4], force sensing [5], robotics surgery [6], etc. However, the translation and rotation motion are coupled in the Stewart mechanism. For many applications, only three degrees-of-freedom (DOF) pure translation or 3DOF pure rotation is needed. This requires only three actuation inputs with less complex kinematics and dynamics analysis than the general 6DOF case. Pure translational parallel mechanisms [7,8] have straight forward kinematics solutions and some commercialized ones have been widely used for fast packaging and haptics [9]. Parallel mechanisms with pure rotation motion have been investigated a lot due to their relatively complex kinematics properties [10-13]. However, traditional parallel mechanisms have only fixed mobility and can be used for one type of motion requirement only. Recent work on parallel mechanisms showed that a parallel mechanism with special designed joints and limbs could reconfigure into different motion types [14-17]. This shows a good advantage in variable applications with lower mobility requirements by using a parallel mechanism with reconfiguration which helps increase precision, simplify control, save energy, and reduce risk by removing extra DOFs.

The 3-(rT)MMP in this paper originated from the study of a class of MPMs based on a patented reconfigurable Hooke (rT) joint [18]. By introducing reconfigurable joints into parallel mechanisms, geometric constraints from limbs can be reconfigurable enabling motion type change of the platform [19]. Based on this principle, the 3-(rT)MPM can reconfigure into pure translation or pure rotation motion. This will be useful for some applications on variable motion tasks, like manufacturing a part with flat surfaces using the pure rotation motion. Another application can be human joint rehabilitation in which the pure translation motion will be used to provide guide to the hand for the arm movement, and the pure rotation motion can be useful for specific joint motion training, like one of those in the shoulder. Similar to the rT joint, a vA joint was also invented to have an MPM with motion change between pure translation and pure rotation [20]. By designing a reconfigurable universal joint, a 3-cylindrical-universal-prismatic joints (CUP) parallel mechanism can also reconfigure into those two motion types [21]. Another way to have this reconfiguration is through constraint singularity with bifurcated motion as illustrated in Ref. [22] by the 3-universal-revolvate-universal joints (URU) parallel mechanism. In addition to the mobility change, the workspace of the pure translation phase and the rotation center of the pure rotation phase of the 3-(rT)MPM can be also reconfigurable. The latter was also realized by using a P$\phi$ limb [23] as a central strut of parallel mechanisms. However, how to model the kinematics...
property change with the reconfiguration has not been investigated much [24,25] and it is worth exploring to fully understand the reconfiguration for mechanism optimization in variable applications.

In parallel mechanism optimal design, the condition number of the Jacobian matrix is generally used to describe kinematics performance in the workspace. Due to the coordinate inhomogeneity and some defective cases of the condition number with parallel mechanisms[26,27], motion/force transmission which represents the effective work of a unit wrench on a unit output twist has been recently well developed and corresponding indices were proposed [26–29]. In Ref. [26], similarity between the motion/force transmission index and the condition number of a dimensionally homogeneous Jacobian of the 3-revolute-prismatic-spherical joints (RPS) parallel mechanism has been found. In this paper, detailed wrench and twist screws are derived for motion/force transmission of both translation and rotation motion of the 3-(rT)P(rT) MPM. It is found that the two factors of the Jacobian matrix determinant represent the output transmissibility and constraint transmissibility. While in the pure translation case the two are different, they are the same in the pure rotation case.

This paper is arranged as follows: Section 2 introduces the reconfigurable joint, limb, and geometric structure of the MPM, its reconfiguration and controllable joint axes. Analytical kinematics analysis is presented in Sec. 3, while Sec. 4 investigates the motion/force transmissibility of two different motion types using screw theory. Following this, Sec. 5 illustrated the variable singularity loci and transmission indices when the joint axes are changed in an example. Conclusions are made in Sec. 6.

2 Geometric Constraint and Reconfiguration of the 3-(rT)P(rT) MPM

2.1 Reconfiguration of the (rT)P(rT) Limb. The key component in the 3-(rT)P(rT) MPM is the reconfigurable Hooke (rT) joint [30], which comprises a housing ring with a groove inside and a bracket as in Fig. 1. The bracket is rigidly attached to link \( b \) and holds axis 1 as the bracket axis. The housing ring is rigidly attached to link \( g \) and holds axis 2 as a radial axis. Inside the housing ring, the radial axis can alter its direction by rotating freely along the groove. This enables the (rT)P(rT) limb to be reconfigurable by changing the direction of the radial axis as shown in Fig. 2.

The (rT)P(rT) limb has a prismatic joint with two reconfigurable rT joints at the two ends, and the ring parts of the rT joints are connected to the base and platform, respectively. For convenience, the rT joint connected to the base is called the base rT joint and the other is called the platform rT joint. By combining radial axes of the two rT joints in each limb to be intersecting or parallel, two working phases are defined as in Fig. 2. In Fig. 2(a), twist screws are associated to each joint axis, while \( S_1 \) and \( S_2 \) are for the radial axes, \( S_3 \) and \( S_3 \) are for the bracket axes, and \( S_4 \) is for the prismatic joint. In the intersecting phase in Fig. 2(a), the two radial axes \( S_1 \) and \( S_3 \) intersect at one point in the plane formed by \( S_1 \) and \( S_4 \), while the two bracket axes \( S_2 \) and \( S_3 \) are parallel. By tuning the radial axes freely, the intersecting point can be along any point on the line. Two different configurations (dashed lines) are shown in Fig. 2(a) in which \( z_4 \) is the angle between the radial axis \( S_3 \) of the platform rT joint and the intersecting line. This angle will be used to define different configurations in the geometric constraint analysis in the following. Similarly, in the parallel phase in Fig. 2(b), the two radial axes \( S_1 \) and \( S_3 \) are tuned to be parallel to each other and have the same angle \( z_4 \) with the defined line. By altering \( S_1 \) and \( S_3 \) at the same time, different parallel phases (for example, \( z_4 \), blue-dashed line in Fig. 2(b)) can be obtained.

Based on the above, the two phases of the (rT)P(rT) limb can make the 3-(rT)P(rT) MPM be reconfigurable between pure rotation and pure translation motion. More than that, in each motion type, the workspace can be tunable by altering the radial axes into different directions as investigated below.

2.2 Reconfiguration of the 3-(rT)P(rT) MPM. The 3-(rT)P(rT) MPM consists of a base, a platform, and three (rT)P(rT) limbs as in Fig. 3. The three limbs support the platform symmetrically around a reference circle of radius \( r_b \), and connect to the base around a circle of radius \( r_p \). For the pure rotation case, the intersecting phase of the limb is used and all the radial axes of the rT joints in the three limbs intersect at point \( o \) as in Fig. 3 in which the three limbs are numbered as limb 1, limb 2, and limb 3. Let \( B_i \) in the base denote the center of the base rT joint and \( A_i \) denote the center of the platform rT joint in the \( i \)th \((i = 1, 2, 3)\) limb and \( A_0 \) the geometric center of the platform. Locate a global coordinate frame \( oxyz \) at the rotation center \( o \) with \( z \)-axis perpendicular to the base and \( y \)-axis in the same plane with line \( ob_1 \). Attach a platform coordinate frame \( o'x'y'z' \) at point \( o' \) with \( z' \)-axis perpendicular to the platform and \( y' \) axis parallel to \( A_iA_0 \) as in Fig. 3(b), where \( o' \) coincides with \( o \) and \( A_0 \) is the center of mass of the platform.

Screw-based analysis [18,31] for the configuration in Fig. 1 showed that each limb provided a constraint force passing through point \( o \) and perpendicular to the plane \( oAB \). Thus, there are three constraint forces intersecting at point \( o \), and three translational DOFs of the moving platform are constrained resulting in that the 3-(rT)P(rT) parallel mechanism has pure rotational DOFs.

In each limb as in Fig. 3, the radial axis of the base rT joint forms an angle \( z_0 \) with \( z \)-axis and is called the base-rT-joint angle, the radial axis of the platform-rT-joint forms an angle \( z_0 \) with \( z' \)-axis and is called the platform-rT-joint angle. Thus, the base radius \( r_b \) and the base-rT-joint angle \( z_0 \) describe the geometric
structure of the base while the platform radius $r_b$ and the platform-rT-joint angle $a_a$ represent the geometric structure of the platform. As the radial axis of the rT joint can be rotated about the bracket axis by any angle, directions of the radial axes $S_{i1}$ and $S_{i5}$ in each limb can be altered freely, hence the intersecting point $o$ can be moved along $z$-axis as in Fig. 4.

Assuming the intersecting point is $o$ when the platform is at the initial configuration, by rotating the radial axes of the rT joints it is changed from $o$ to $o'$. Then, the base-rT-joint angle $a_b$ and platform-rT-joint angle $a_a$ are changed to $a_b'$ and $a_a'$, respectively, the constraint equation among these parameters can be obtained as

$$
\pm \Delta h = r_b \cot z_0 - r_a \cot z_0' = r_a \cot z_a - r_a \cot z_a' \tag{1}
$$

where $\Delta h$ is the distance between $o$ and $o'$. When the change from $o$ to $o'$ is to the positive direction of $z$-axis, the sign before $\Delta h$ is “+,” otherwise it is “−.”

After assembling the parallel mechanism, the initial angles $z_0$ and $z_a$ are known, a desired configuration of the 3-(rT)P(rT) parallel mechanism can be obtained by calculating (1) when a needed moving distance $\Delta h$ of rotation center $o$ or rT joint angle $z_0'$ or $z_a'$ is given. Thus, by altering the rT joint to change the position of the intersecting point $o$ between negative infinite and positive infinite of $z$-axis, the platform can rotate about any point on the $z$-axis. Hence, the rotation center of the 3-(rT)P(rT) parallel mechanism can be controlled according to the requirement, and the corresponding rotational workspace can be controllable.

Based on the configuration in Fig. 5, when altering all the limbs to the parallel phase in which the radial axes ($s_{i1}$ and $s_{i5}$) of the two rT joints in each limb are parallel, each limb will provide a constraint moment to the platform and the mechanism will have pure translation motion \[ 18\]. Based on the parallel phase of the limb as in Fig. 2(b), the mechanism will keep the pure translation motion but with variable workspace and performance when tuning the radial axes to be parallel in different directions represented by different angle $z_a$. This will be explored more in the motion/force transmissibility section. In this phase, the setting of the coordinate frames follows that in the pure rotation phase in Fig. 2 but the center of the platform coordinate frame $o'$ will be different with point $o$ due to the platform translation.

3 Kinematics Analysis of the 3-(rT)P(rT) MPM With Pure Translation and Pure Rotation

3.1 Geometric Parameters and Inverse Kinematics. Let $a_i$ and $b_i$ denote the position vectors of the platform rT joint center $A_i$ and the base rT joint center $B_i$, respectively, expressed in the global coordinate frame $oxyz$. Let $a_i'$ denote the platform-rT-joint position vectors of center $A_i$ expressed in the platform coordinate frame $ox'y'z'$, and $d_i$ and $d_i'$ denote the length and limb translation
vector from point $B_i$ to $A_i$ in the global coordinate frame $oxyz$. Based on these settings and the geometric structure of the mechanism, the base-$rT$-joint position vectors can be determined as

$$s_i = [\cos(3\pi/2 + 2\pi(i-1)/3), \sin(3\pi/2 + 2\pi(i-1)/3), -\cot\theta_i]^T,$$

$$b_i = r_i s_i, \quad (i = 1, 2, 3) \tag{2}$$

where $0 < \theta_i < \pi, \theta_i$ represents the unit vector of the joint axis $j$ in limb $i$.

The platform-$rT$-joint position vectors can be expressed in the platform coordinate frame as

$$s_3 = [\cos(3\pi/2 + 2\pi(i-1)/3), \sin(3\pi/2 + 2\pi(i-1)/3), \cot\theta_i]^T,$$

$$a_i' = r_i s_i, \quad (i = 1, 2, 3) \tag{3}$$

where $0 < \theta_i < \pi$.

Let $R$ be the $3 \times 3$ rotational matrix and $p$ be the translation vector of the moving coordinate frame with respect to the global coordinate frame, the closed-loop equation of each limb can be expressed in the global coordinate frame as

$$a_i = b_i + d_i s_3 = p + R a_i' \quad (i = 1, 2, 3) \tag{4}$$

The inverse kinematics is to get input $d_i$, which can be obtained directly from Eq. (4)

$$\sqrt{(R a_i' + p - b_i)^T (R a_i' + p - b_i)} = d_i \quad (i = 1, 2, 3) \tag{5}$$

For the pure rotation phase, there is no translation and $p$ equals to zero in Eq. (5). When $R$ is known, the limb lengths can be solved from Eq. (5). For the pure translation phase, rotation matrix $R$ will be the identity matrix. The inverse kinematics can be easily solved from Eq. (5) when giving the platform position $p$.

### 3.2 Forward Kinematics

In the forward kinematics analysis, limb length $d_i$ will be known and rotation matrix $R$ in the pure rotation phase or translation vector $p$ in the pure translation phase will be solved. For the pure translation case, $p$ can be solved from Eq. (4)

$$p = b_i + d_i s_3 - a_i' \quad (i = 1, 2, 3) \tag{6}$$

The forward kinematics of the pure rotation case is more complex and can be solved in the following way. Setting $p$ equal to zero in Eq. (5) and expanding it gives

$$2 (R a_i')^T b_i = (r_{ij} / \sin \theta_i)^3 + (r_{ij} / \sin \theta_i)^3 - d_i^2$$

$$= 2 r_{ij} r_{1b} \cos(\phi_i) / (\sin \theta_i \sin \theta_i) \quad (i = 1, 2, 3) \tag{7}$$

which can be described as

$$(R s_3)^T s_3 = \cos(\phi_i) \quad (i = 1, 2, 3) \tag{8}$$

where $\phi_i$ is the angle between axes $s_3$ and $s_5$, see Fig. 3.

In the three-dimensional space, a rotation matrix $R$ between two coordinate systems can be given by Cayley’s formula [32]

$$R = \nabla^{-1} \begin{bmatrix} 1 + c_i^2 - c_j^2 - c_k^2 & 2(c_i c_j - c_k) & 2(c_i c_k + c_j) \\ 2(c_i c_j + c_k) & 1 - c_i^2 + c_j^2 - c_k^2 & 2(c_j c_k - c_i) \\ 2(c_k c_j - c_i) & 2(c_j c_k + c_i) & 1 - c_i^2 - c_j^2 + c_k^2 \end{bmatrix}$$

where $\nabla = 1 + c_i^2 + c_j^2 + c_k^2, c_i, c_j, c_k$ and $c_3$ are the Rodrigues–Hamilton parameters [32].

Substituting Eq. (9) into Eq. (8), simplifying and taking the numerators, gives

$$f_i(1, c_1, c_2, c_3, c_1 c_2, c_1 c_3, c_2 c_3, c_1^2, c_2^2, c_3^2) = 0 \quad (i = 1, 2, 3) \tag{10}$$

where $f_i(\cdot)$ is a function of the unknown power products in the bracket, with real constant coefficients depending on the input and mechanism dimension parameters only.

Rewrite Eq. (10) as the function of $c_1$ and $c_2$ by putting the products of power in $c_3$ into the coefficients

$$Q_{e-jc_1} c_1 = \sum_{i+j \leq 2, i \neq j} G_{e-jc_1} c_i c_j = 0 \quad (e = 0, 1, 2) \tag{11}$$

where $G_{e-jc_1}$ are functions of $g_{e-jc_1}$ and power products of $c_3$.

Construct the following matrix:

$$\Delta(c_1, c_2, t_1, t_2) = \begin{bmatrix} Q_0(c_1, c_2) & Q_1(c_1, c_2) & Q_2(c_1, c_2) \\ Q_0(t_1, t_2) & Q_1(t_1, t_2) & Q_2(t_1, t_2) \\ Q_0(t_1, t_2) & Q_1(t_1, t_2) & Q_2(t_1, t_2) \end{bmatrix}$$

where $t_1$ and $t_2$ are the intermediate parameters only. By developing the above equation, $\Delta(c_1, t_1, t_2)$ is a polynomial of order 2 in $c_1$, 3 in $c_2$, 3 in $t_1$, and 2 in $t_2$, Dixon observed that $\Delta$ vanishes when $t_1$ and $t_2$ substitute for $c_1$ and $c_2$, implying that $(c_1 - t_1)(c_2 - t_2)$ is a factor of $\Delta$. Therefore, the expression

$$\delta(c_1, c_2, t_1, t_2) = \frac{\Delta(c_1, c_2, t_1, t_2)}{(c_1 - t_1)(c_2 - t_2)} = 0 \quad (13)$$

is a polynomial of order 1 in $c_1$, 2 in $c_2$, 2 in $t_1$, and 1 in $t_2$. $\delta$ vanishes when $Q_0(c_1, c_2), Q_1(c_1, c_2),$ and $Q_2(c_1, c_2)$ have common zeros no matter what $t_1$ and $t_2$ are. The coefficients of each power product $t_1^j t_2^h$ $(i = 0, 1, 2; j = 0, 1)$ of $\delta$ have common zeros which are also the common zeros of equations $Q_0, Q_1,$ and $Q_2$ in Eq. (11). This gives five equations in power product of $c_1$ and $c_2$, whereas the numerator of the power product $c_1^j c_2^h$ $(i = 0, 1; j = 0, 1, 2)$ is also five. Therefore, the coefficients of each power product $c_1^j c_2^h$ in these five equations form a $5 \times 5$ matrix $D$. All the above algorithm can be expressed as

$$\delta(c_1, c_2, t_1, t_2) = \frac{\Delta(c_1, c_2, t_1, t_2)}{(c_1 - t_1)(c_2 - t_2)} = T D C^T = 0 \tag{14}$$

where $T = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1 t_2 & t_2 \\ 1 & c_2 & c_2^2 & c_1 & c_2 c_1 \end{bmatrix}$, and $D$ is a matrix whose elements are polynomials in $c_1 c_2$.

Equation (11) has common zeros if the determinant of the matrix $D$ equals to 0. Thus, an equation in $c_3$ can be obtained

$$\sum_{j=0}^{8} h_i c_3^j = 0 \quad (15)$$

where $h_i$ are the real constants depending on input data only.

This implies that an univariate equation in $c_3$ of order 8 is obtained. Solving Eq. (15), all the solutions for $c_3$ can be obtained. Then substitute $c_3$ into the following equation:

$$D C^T = 0 \tag{16}$$

According to the Cramer’s rule, the solutions of $c_1$ and $c_2$ can be computed from the above linear system. Substituting all the solutions of $c_1$, $c_2$, and $c_3$ into Eq. (9), orientation $R$ can be obtained and the forward kinematics is solved.

### 4 Motion/Force Transmissibility and Singularity Representation

Motion/force transmission shows work of a wrench on a twist. Three different transmission types with corresponding indices were introduced including input transmission, output
transmission, and constraint transmission [28]. For the \((rT)P(rT)\)
limb, the input transmission, defined as reciprocal product of the
input twist screw of the actuator and its transmission wrench screw
in a limb, is constant as the actuation wrench is in the same
line with the actuation twist which is along the prismatic joint.
Their reciprocal product gives 1 and it is not considered further.
Here, transmission wrench screw represents a wrench by which
motion/force from the actuator is transmitted to the moving plat-
form. The other two transmission indices will be investigated for
the pure translation and pure rotation topologies which have dif-
ferent transmission formats.

4.1 3-(rT)P(rT) With Pure Translation. In the pure transla-

tion case, all the limbs in the parallel phase as in Fig. 2(b).

Based on the five twist screws [33] associated to the five 1DOF
joint axes, the transmission wrench screw can be obtained by
locking the actuated joint and taking the reciprocal screw to all
other twist screws except the locked one as

\[
S_{Ti} = \left[ s_{i3}^T \times s_{i3}^T \right] \quad (i = 1,2,3)
\]  
(17)

which is a force along the prismatic joint and passing by the plat-
form \(rT\) joint center \((a)\) that represents the actuation input.

By taking the reciprocal screw to all the five twist screws in

Fig. 2, the constraint wrench screw in each limb can be obtained

\[
S_{CI} = \left[ 0 \times s_{i3}^T \right] \quad (i = 1,2,3)
\]  
(18)

which is a moment in the direction perpendicular to the two rota-
tion axes \(s_{j3}\) and \(s_{k3}\) of the platform \(rT\) joint in limb \(i\), where \(\theta\)

is the zero vector \([0,0,0]\), \(s_{i3} = s_{i3} \times s_{i3}/|s_{i3} \times s_{i3}|\).

Thus, the screw-based overall Jacobian matrix can be directly
obtained and its determinant equals to

\[
 [J] = \begin{bmatrix}
 S_{T1} \\
 S_{T2} \\
 S_{T3} \\
 S_{T2} \\
 S_{T3} \\
 S_{T3} \\
 S_{T2} \\
 S_{T3}
 \end{bmatrix} = \begin{bmatrix}
 s_{i3}^T \times s_{i3}^T \\
 s_{i3}^T \times s_{i3}^T \\
 0 \times s_{i3}^T \\
 0 \times s_{i3}^T \\
 0 \times s_{i3}^T \\
 0 \times s_{i3}^T \\
 0 \times s_{i3}^T \\
 0 \times s_{i3}^T
 \end{bmatrix} \begin{bmatrix}
 s_{i3}^T \\
 s_{i3}^T \\
 s_{i3}^T \\
 s_{i3}^T \\
 s_{i3}^T \\
 s_{i3}^T \\
 s_{i3}^T \\
 s_{i3}^T
 \end{bmatrix}
\]  
(19)

which represents the singularity configurations of the parallel
mechanism when it equals to zero. Later analysis will show the
direct relation between the motion/force transmission and this
determinant.

4.1.1 Output Transmission. It is defined as a reciprocal pro-
duct of the output motion twist screw of the platform and the trans-
mission wrench screw of a limb actuator. By locking two limbs
except the \(i\)th limb, the platform will have a 1DOF motion. Its
twist screw \(S_{Oi}\) is called the output twist screw and can be
obtained by taking the reciprocal screw to the other five wrench
screws in the Jacobian matrix in Eq. (19) except the \(i\)th transmis-
sion wrench \(S_{Ti}\)

\[
S_{Oi} = \left[ 0 \times s_{j3}^T \times s_{k3}^T / |s_{j3} \times s_{k3}| \right] \quad (i,j,k = 1,2,3; i \neq j \neq k)
\]  
(20)

which is a pure translation along the line that is perpendicular to
both limb \(j\) and limb \(k\).

Then, the output transmission virtual coefficient [34] is repres-
ented by the reciprocal product

\[
S_{Ti} \cdot S_{Oi} = s_{i3} \cdot s_{j3} \times s_{k3} / |s_{j3} \times s_{k3}|
\]  
(21)

which shows the work of the actuation input of limb \(i\) on the
1DOF translation motion of the platform when the other two limbs
are locked. Based on this, the power coefficient is defined as

\[
\dot{\lambda}_{Ti} = \frac{|S_{Ti} \cdot S_{Oi}|}{|S_{Ti} \cdot S_{Oi}|_{\text{max}}} = \frac{\cos \theta_i}{1}
\]  
(22)

where \(\theta_i\) is the angle between limb \(i(s_{i3})\) and the line \(s_{j3} \times s_{k3}\),
which is perpendicular to both the other two limbs.

Thus, there are three power coefficients for the three limbs and
they depend on the limb directions. The minimum power coeffi-
cient is taken as the output transmission index (OTI)

\[
\gamma_i = \min\{\dot{\lambda}_{Ti}, \dot{\lambda}_{T2}, \dot{\lambda}_{T3}\} = \min\{\cos \theta_1, \cos \theta_2, \cos \theta_3\}
\]  
(23)

which can be used to represent the output transmission perform-
dance at the given mechanism configuration.

The output transmission represents the contribution of the limb
to the platform motion. When any \(\theta_i\) is \(\pi/2\), the virtual coefficient
becomes zero indicating that limb \(i\) cannot transmit any power to
the platform along its motion. This represents singular configura-
tion of the parallel mechanism. It can be also noted that

\[
s_{i3} \cdot s_{j3} \times s_{k3} = s_{i3} \cdot s_{j3} \times s_{k3} = s_{i3} \cdot s_{j3} \times s_{k3} = 0
\]  
(24)

which means that the three limb power coefficients represent the
same singularity configurations and it also equals to the first factor
of the Jacobian matrix determinant in Eq. (19).

4.1.2 Constraint Transmission. Similar to the output transmis-
sion, constraint transmission is the reciprocal product of the
virtual output motion twist screw of the platform and the con-
straint wrench screw of a limb and can be obtained in the follow-

ing way. Locking all the limb actuation and releasing the
constraint \((S_{Ci})\) from limb \(i\), the platform can virtually have a
1DOF twist motion \(S_{OCI}\), which can be obtained by taking the
reciprocal screw to the other five wrench screws in the Jacobian
matrix in Eq. (19) except the \(i\)th constraint wrench \(S_{CI}\)

\[
S_{OCI} = [s_{i3}^T \times s_{i3}^T]_{\text{OCI}}
\]  
(25)

which is a pure rotation motion along the line that is perpendicular
to both \(s_{i3}\) in limb \(j\) and \(s_{i3}\) in limb \(k\), where \(s_{OCI} = s_{OCI} \times s_{OCI} / |s_{OCI} \times s_{OCI}|\), \((i,j,k = 1,2,3; i \neq j \neq k)\), and \(r_{OCI}\)

represents a point on the line \(s_{OCI}\).

Then, the constraint transmission virtual coefficient is obtained
by the reciprocal product

\[
S_{CI} \cdot S_{OCI} = s_{i3} \cdot s_{OCI} = \cos \theta_{CI}
\]  
(26)

which shows the work of the constraint wrench \(S_{CI}\) on the virtual
1DOF motion of the platform. \(\theta_{CI}\) is the angle between \(s_{OCI}\)
and \(s_{OCI}\).

Following this, the power coefficient is defined as:

\[
\dot{\lambda}_{CI} = \frac{|S_{CI} \cdot S_{OCI}|}{|S_{CI} \cdot S_{OCI}|_{\text{max}}} = \frac{\cos \theta_{CI}}{1}
\]  
(27)

and the minimum one is taken as the constraint transmission index
(CTI)

\[
\gamma_{CI} = \min\{\dot{\lambda}_{CI}, \dot{\lambda}_{C2}, \dot{\lambda}_{C3}\} = \min\{\cos \theta_{C1}, \cos \theta_{C2}, \cos \theta_{C3}\}
\]  
(28)

which can be used to represent the closeness to constraint
singularity.

When Eq. (26) equals to zero, the constraint wrench cannot pro-
vide any constraint to the defined motion of the platform which
will gain extra mobility. Thus, the mechanism meets constraint
singularities which can be expressed as

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which shows that the three constraint power coefficients represent the same constraint singularity configurations of the platform and it equals to the second factor of the Jacobian matrix determinant in Eq. (19).

Thus, by taking the Jacobian matrix determinant in Eq. (19) equal to zero, all the output transmission singularities and constraint singularities of the 3-(rT)P(rT) MPM with pure translation motion can be expressed.

4.2 3-(rT)P(rT) With Pure Rotation. In the pure rotation case, all the limbs are in the intersecting phase as in Fig. 2(a). The transmission wrench screw comes from the prismatic joint in each limb and it is the same with that in Eq. (17) that is a force along the prismatic joint. The constraint wrench can be obtained in the same way for Eq. (18), but it becomes a constraint force in this pure rotation case

\[ S_{ij} = [s_i^T \ 0] \quad (i = 1,2,3) \]  

(30)

which is a force parallel to the bracket axis (s_{ij}) of the rT joint and passing by the rotation center o.

Similarly, the screw-based overall Jacobian matrix J can be directly obtained and its determinant equals to

\[
\begin{bmatrix}
S_{T1} \\
S_{T2} \\
S_{T3} \\
S_{12} \\
S_{13} \\
S_{23}
\end{bmatrix}
= 
\begin{bmatrix}
\left[ s_{13}^T \ a_i^T \times s_{13}^T \right] \\
\left[ s_{12}^T \ a_i^T \times s_{12}^T \right] \\
\left[ s_{13}^T \ a_i^T \times s_{13}^T \right] \\
\left[ s_{12}^T \ 0 \right] \\
\left[ s_{13}^T \ 0 \right] \\
\left[ s_{12}^T \ 0 \right]
\end{bmatrix}
\]

\[
= m_i m_j m_k
\]

(31)

where m_i is the magnitude of a_i \times s_{ij}. Equation (31) represents singularity configurations of the 3-(rT)P(rT) MPM with pure rotation motion when it equals to zero. It can be seen that it fully depends on the relation among bracket axes (s_{12}, s_{23}, s_{31}) of the rT joints in the three limbs.

4.2.1 Output Transmission. Similarly by locking two limbs except the ith limb, the platform will have a 1DOF motion. The output twist screw S_{Tj} is obtained by taking reciprocal screw to the other five wrench screws in the Jacobian matrix in Eq. (31) except the jth transmission wrench S_{Tj}

\[ S_{Oj} = \left[ s_{ij}^T \left/ s_{ij} \times s_{ij} \right. \right] \quad (i,j,k = 1,2,3; i \neq j \neq k) \]  

(32)

which is a pure rotation along the line passing by the rotation center o and perpendicular to both bracket axes in limb j and limb k.

Then, the output transmission virtual coefficient is expressed by the reciprocal product

\[ S_{Tj} \cdot S_{Oj} = m_i s_{ij} \times s_{ij} / \left/ s_{ij} \times s_{ij} \right. \]  

(33)

which shows the work of the actuation input of limb i on the 1DOF rotation motion of the platform when the other two limbs are locked.

Based on this, the power coefficient is defined as

\[ \lambda_{Tj} = \frac{\left| S_{Tj} \cdot S_{Dj} \right|}{\left| S_{Tj} \cdot S_{Dj_{\max}} \right|} = \frac{m_i \cos \beta_i}{m_i + 1} = \cos \beta_i \]  

(34)

where \( \beta_i \) is the angle between the bracket axis (s_{ij}) in limb i and the line (s_{ij} \times s_{ij}), which is perpendicular to both bracket axes of the other two limbs.

Thus, OTI of the pure rotation case is

\[ \gamma_{OTI} = \min \{ \lambda_{T1}, \lambda_{T2}, \lambda_{T3} \} = \min \{ \cos \beta_1, \cos \beta_2, \cos \beta_3 \} \]  

(35)

which is used to describe the closeness to the output transmission singularity. When any of them equals to zero, the mechanism meets singularity and the limb cannot transmit power to the platform. It is also noted that

\[ s_{12} \cdot s_{22} \times s_{32} = s_{22} \cdot s_{12} \times s_{32} = s_{32} \cdot s_{12} \times s_{22} = \begin{bmatrix} s_{12}^T \\ s_{22}^T \\ s_{32}^T \end{bmatrix} = [J] = 0 \]  

(36)

which shows that the zero output transmission power coefficients are equal to each other and also equal to the zero Jacobian matrix determinant. They represent the same singularity configurations of the 3-(rT)P(rT) MPM with pure rotation motion.

4.2.2 Constraint Transmission. Using similar way with the pure translation case, by locking all the limb actuation and releasing the constraint (S_{Cu}) from limb i, a 1DOF platform twist motion S_{OCu} can be obtained by taking reciprocal screw to the other five wrench screws in the Jacobian matrix in Eq. (31) except the ith constraint wrench S_{Cu}

\[ S_{OCu} = \left[ \left[ r_{OCu} \times s_{OCu} \right] \right] = \left[ s_{OCu} \right] = m_j s_{ij} \times s_{ij} \]  

(37)

which is a pure rotation motion along the line s_{OCu}, and passing by a point r_{OCu}. Since it is reciprocal to the other two constraint forces which pass by the origin o, the second part of this twist S_{OCu} should have r_{OCu} \times S_{OCu} = m_j s_{ij} \times s_{ij} which is perpendicular to both the other two constraint forces represented by bracket axes of the other two limbs. m_j denotes the magnitude.

Following this, the constraint transmission virtual coefficient is calculated by the reciprocal product:

\[ S_{Cu} \cdot S_{OCu} = m_j s_{ij} \times s_{ij} = m_j \cos \beta_i \]  

(38)

which shows the constraint work of the constraint wrench S_{Cu} on the virtual 1DOF motion of the platform. In general, the platform motion is constrained when Eq. (38) is not zero and the mechanism meets constraint singularity if it becomes zero.

Following this, the constraint power coefficient is:

\[ \lambda_{Cu} = \frac{\left| S_{Cu} \cdot S_{OCu} \right|}{\left| S_{Cu} \cdot S_{OCu_{\max}} \right|} = \frac{m_j \cos \beta_i}{m_j + 1} = \cos \beta_i = \lambda_{Tj} \]  

(39)

which is the same with the output transmission power coefficient in Eq. (34). Thus, the constraint transmissibility is the same with the output transmissibility of the 3-(rT)P(rT) MPM with pure rotation motion. This can also be seen from the Jacobian matrix determinant in Eq. (31) which has square of the same factor of virtual coefficient of the output transmission and the constraint transmissibility. Then, the CTI is the same with the OTI in Eq. (35).

Based on the above, it can be concluded that the factors of the Jacobian matrix determinant of the 3-(rT)P(rT) MPM represent its output transmissibility and constraint transmissibility. While in the pure translation case the two are different as in Eq. (19), they are the same in the pure rotation case as in Eq. (31). When the Jacobian determinant equals to zero, output transmission and constraint singularities can be found.
5 Variable Motion/Force Transmissibility and Singularity Loci

As explained in Sec. 2, by tuning the radial axes of the rT joints in all the limbs, the rotation center of the pure rotation motion can be controlled and the workspace of the pure translation motion can be also controllable. At the same time, their transmissibility and singularity loci will be variable. To demonstrate this, four different configurations are selected as in Fig. 6 and represented by angle \( \alpha_{ai} \) which is the angle between the radial axis of the platform rT joint and the \( z \)-axis. At those four configurations, the radial axis of the platform rT joint intersects with the \( z \)-axis at point \( P_i \) \( i = 1, 2, 3, 4 \). As in Fig. 6, \( P_1 \) is above the platform, \( P_2 \) is on the platform, \( P_3 \) is in the middle, while \( P_4 \) is under the base. Correspondingly, \( \alpha_{a1} = 2\pi/3 \), \( \alpha_{a2} = \pi/2 \), \( \alpha_{a3} = \sin^{-1}(\sqrt{2}/3) \), and \( \alpha_{a4} = \pi/6 \). In the following examples, the platform and base sizes are set \( r_a = 1 \) and \( r_b = 2 \).

5.1 3-(rT)P(rT) With Pure Translation. Variable transmission indices and singularity loci of the 3-(rT)P(rT) MPM with pure translation are illustrated in Fig. 7 corresponding to the four different radial axis setup of the rT joint in Fig. 6. Based on Eqs. (21)–(23), the OTI depends only on the directions of the three limbs and is the same as in Fig. 7(e) for all the four cases. When the OTI equals to zero, output transmission singularity occurs for the 3-(rT)P(rT) MPM which are represented by the same yellow plane \( z = 0 \) in all the four cases in Figs. 7(a)–7(d). This shows that the tuning of the radial axes of the rT joints will not affect the OTI and singularity in the pure translation motion. Differently, the radial axes change will bring different CTIs (right: blue curves) and constraint singularity loci (left: blue surfaces) as shown Figs. 7(a)–7(d). In general, the transmission index values are high when the platform is close to the \( z \)-axis \( (x = y = 0) \) and it decrease when the platform moves away. A special case is that when the radial axes are set to be perpendicular to the \( z \)-axis with \( \alpha_{a2} = \pi/2 \) as in Fig. 7(b), the constraint transmission singularity loci coincide with the output transmission singularity loci which are the plane \( z = 0 \). This means that all the areas above or below this plane are singularity-free workspaces.

Comparing the CTIs for the four cases, it can be seen that workspace close to \( z \)-axis with high index values is larger when the radial axes of the platform rT joints have smaller angle with the \( z \)-axis. A big area with CTI = 0.9 has been shown in Fig. 7(d) and it is close to the output transmission performance in Fig. 7(e).

![Fig. 6 Four different configurations](image)

![Fig. 7 Variable transmission indices and singularity loci of the pure translation motion: (a) \( \alpha_{a1} = 2\pi/3 \) (left: singularity loci and right: CTI at \( z = 1.5 \)), (b) \( \alpha_{a2} = \pi/2 \) (left: singularity loci and right: CTI at \( z = 1.5 \)), (c) \( \alpha_{a3} = \sin^{-1}(\sqrt{2}/3) \) (left: singularity loci and right: CTI at \( z = 1.5 \)), (d) \( \alpha_{a4} = \pi/6 \) (left: singularity loci and right: CTI at \( z = 1.5 \)), and (e) OTI at \( z = 1.5 \)](image)
Sec. 5.1, workspace close to $c_3$-axis ($c_1 = c_2 = 0$) with high index values is larger when the rotation center is close to the negative side of the $z$-axis. A big area with CTI/OTI over 0.8 is shown in Fig. 8(d) for the case that the rotation center is below the base represented by point $P_4$ in Fig. 6.

6 Conclusions
This paper presented the reconfiguration of a 3-(rT)rT MPM between the pure translation (3T) and pure rotation (3R) motion. The reconfiguration was based on a reconfigurable Hooke (rT) joint which enabled the 3-(rT)rT MPM to not only have the reconfiguration but also with controllable workspace for both cases. Based on the motion/force transmissibility investigation, it was found that the two factors of the Jacobian matrix determinant of the 3-(rT)rT MPM represented its output transmissibility and constraint transmissibility. While in the pure translation case, the two are different, they are the same in the pure rotation case. The zero output transmissibility and CTIs represented singularity configurations of the parallel mechanism. Numerical examples demonstrated the controllable workspace change with variable transmissibility and singularity loci for both pure rotation and pure translation. It was concluded that when the intersecting point between the radial axes of the platform rT joints and the $z$-axis was closer to the negative side of $z$-axis, the singularity-free workspace with good index was bigger. Later motion/force transmissibility indices will be used to optimize the basic design parameters covering the 3T and 3R motion. Future work will also consider dynamics, control, and trajectory planning considering the two different motion types.

References


