Abstract—Modeling and control of soft robots is an up-to-date and exciting area of research which has been tackled with complementary approaches so far. In this paper, we modify the existing continuum Cosserat approach optimizing it for soft robot arms which can be discretized in a finite number of sections and degrees of freedom. The resulting new piece-wise constant strain model extends the existing piece-wise constant curvature model by allowing torsion and shears strains which are fundamental to cope with out-of-the-plane external forces as appearing for example during ground locomotion. A first experimental comparison has been also conducted using one fluidic actuated leg of the soft crawler FASTT.

I. INTRODUCTION

Soft robotics is becoming rapidly an established chapter in the robotics field, with many researchers contributing to the challenges that it introduces [21], [11]. Certainly, modeling and control of such infinite Degrees of Freedom (DoF) robots is one of them. Several approaches have been proposed so far, which have different applications. They may be divided into three main categories: Piece-wise Constant Curvature (PCC) models, continuum Cosserat models and 3D Finite Elements Models (FEM).

The PCC models represent the soft robot as a finite collection of circular arcs, which can be described by only three parameters (radius of curvature, angle of the arc and bending plane), drastically reducing the number of variables needed. Originally devoted to kinematics modeling [10], it has been extended and improved over the years with excellent results [9], [7]. On the other hand, the constant curvature assumption is not always valid, especially under non-negligible external loads including gravity.

The continuum Cosserat approach is an infinite DoF model where the soft robot is represented by the stacking of infinitesimal micro-solids. It has been primarily used in the contest of hyper-redundant robot [5] and more recently it has been applied to soft robotics locomotion [2], [1] and manipulation [20], in both static [16] and dynamic [17] conditions. This method has been also extended to shell-like soft robots for underwater locomotion inspired by cephalopods [18]. Despite their accuracy and fidelity to the continuous mechanics of the soft robots, the resulting partial differential equations are computationally demanding and hard to be applied for control purposes.

Finally, a FEM based approach has also been explored for modeling and real-time control of soft robots [12], which is so far limited to quasi-static conditions and is subject to a linearisation of the structural elasticity that may not apply to many soft robot geometries.

In the effort of developing a dynamic model for soft robotics that is both computationally inexpensive and able to cope with non-negligible external forces, a new Piece-wise Constant Strain (PCS) model is presented here. It is a discretization of the continuous Cosserat method based on the piece-wise constant strain assumption, where, with respect to the PCC model, the PCS model introduces the possibility to have torsion and shear deformations as well, which are essentials to cope with out-of-the-plane external loads. From a geometrical point of view, the circular arcs with fixed curvature constituting the kinematics of a PCC model have been generalized by screw arcs with fixed axis and pitch in the PCS model. Furthermore, the discrete and continuous Cosserat models form a useful unified framework for the modeling of soft robots, linked by analytic exponential maps.

The model presented in this work can be highly beneficial for describing the behaviour under external influences of the Bending Fluidic Actuators (BFAs), which are receiving high attention in soft robotics due to their robustness, smooth interaction with the environment, high force density and large deformations. They have been applied in various scenarios, including manipulation [13], rehabilitation [14] and locomotion [24]. A recent soft robot designed for ground locomotion with BFAs, called FASTT [4], is shown in figure 1. This robot exploits the compliance of the BFAs to obtain self-stabilized locomotion on both flat and uneven terrains. It also features variable morphology, making it able to face different external conditions.

In the following, a brief resume of the continuous Cosserat model is presented in section II (for a detailed exposition see [17]), which is preparatory to the development of the new PCS model described in section III. Then, simulation results are shown in section IV, while a single BFA leg of FASTT has been used for a first experimental comparison in section V.

II. CONTINUOUS COSSE RAT MODEL

In the Cosserat theory, the configuration of a micro-solid of a soft body with respect to the inertial frame at a certain time is characterized by a position vector \( u \) and a material orientation matrix \( R \), parameterized by the material abscissa.
X ∈ [0, L] along the robot arm. Thus, the configuration space is defined as a curve g(X) ∈ SE(3) with

\[
g = \begin{pmatrix} R & u \\ 0 & 1 \end{pmatrix}.
\]

Then, the strain state of the soft arm is defined by the vector field along the curve g(X) given by \( \hat{\xi}(X) = g^{-1} \partial g / \partial X = g^{-1}g' \in se(3) \), where the hat is the isomorphism between the twist vector representation and the matrix representation of the Lie algebra se(3). The components of this field are specified in the micro-solid frames as:

\[
\hat{\xi} = \begin{pmatrix} \tilde{k} & q \\ 0 & 0 \end{pmatrix} \in se(3), \quad \xi = (k^T, q^T)^T \in \mathbb{R}^6,
\]

where \( q(X) \) represents the linear strains, and \( k(X) \) the angular strains. The tilde is the isomorphism between three dimensional vectors and skew symmetric matrix. The time evolution of the configuration curve \( g(X) \) is represented by the twist vector field \( \eta(X) \in \mathbb{R}^6 \) defined by \( \hat{\eta}(X) = g^{-1} \partial g / \partial t = g^{-1} \dot{g} \). This field can be detailed in terms of their components in the micro-solid frames as:

\[
\hat{\eta} = \begin{pmatrix} \tilde{w} & v \\ 0 & 0 \end{pmatrix} \in se(3), \quad \eta = (w^T, v^T)^T \in \mathbb{R}^6.
\]

where \( v(X) \) and \( w(X) \) are respectively the linear and angular velocity at a given instant.

A. Continuous Kinematics

Given the above construction, we can obtain the kinematic equations relating the strains of the robot arm (\( \hat{\xi}(X) \)) with the position (\( g(X) \)), velocity (\( \eta(X) \)) and acceleration (\( \hat{\eta}(X) \)) of each infinitesimal micro-solid constituting the robot. By definition, the first equation is given by:

\[
g' = g \hat{\xi}.
\]

Then, the equality of mixed partial derivatives \( (g')' = (g')' \) give the following compatibility equation between strain and velocity (recall that \( \dot{g} = g \hat{\eta} \)):

\[
\eta' = \hat{\xi} - \text{ad}_\xi \eta.
\]

Finally, by taking the derivative of (2) with respect to time, we obtain the acceleration continuous equation:

\[
\hat{\eta}' = \hat{\xi} - \text{ad}_\xi \eta - \text{ad}_\xi \hat{\eta}.
\]

B. Continuous Dynamics

In [3] it is shown that the beam (as well as shell and 3D body) dynamics of a Cosserat arm can be directly derived from the extension to continuum media of the Poincaré equations of mechanics by taking a Lagrangian density \( \mathcal{L}(\eta) = \mathcal{L}(\xi) \), being \( \mathcal{L} \) and \( \mathcal{L} \) the densities of kinetic and elastic energy of the Cosserat beam per unit of material length. After some algebra, we obtain the following dynamic equation for one dimensional Cosserat media with respect to the micro-solid frames.

\[
\mathcal{M} \frac{\partial}{\partial t} \eta = \mathcal{F}_1 + \text{ad}_\xi \mathcal{F}_1 + \mathcal{F}_a + \mathcal{F}_e - \text{ad}_\xi (\mathcal{M} \eta),
\]

where \( \mathcal{F}_1 \) is the external wrench of distributed applied forces and \( \mathcal{M} \) the screw inertia matrix. Let us specify the angular and linear components of the internal and external wrenches: \( \mathcal{F}_1 = (M^T, N^T)^T \), \( \mathcal{F}_a = (m_a^T, n_a^T)^T \), \( \mathcal{F}_e = (m_e^T, n_e^T)^T \in \mathbb{R}^6 \), where \( m_0(X) \) and \( M(X) \) are the internal force and torque vectors, \( n_0(X) \) and \( m_0(X) \) are the actuation force and torque input, while \( n_a(X) \) and \( m_a(X) \) are the external force and torque for unit of \( X \). By choosing a local micro-solid frame oriented as in figure 2, with the \( x \) axis pointing toward the tip of the robot arm and the \( y \) and \( z \) axes laying on the plane of the section (considered symmetric), the screw inertia matrix is equal to: \( \mathcal{M} = \text{diag}(1111110) \rho \), where \( \rho \) is the body density, \( A \) is the section area and \( J, I \) are respectively the bending and torsion second moment of inertia of the micro-solid.

III. DISCRETE COSERROT MODEL

Equations (1), (2), (3) and (4) (as well as the ones for bi-dimensional media) of the continuous Cosserat model are suitable to model the kinematics and dynamics of soft robots expressing a non-constant curvature behavior, as has been presented in [17], [18]. In the effort of unifying the constant and non-constant cases under the same mathematical framework, the above model has been applied to the former case by an analytic spatial integration. It results that, in this framework, the piece-wise constant model becomes a case of piece-wise integrability of the continuum model, providing the extension to piece-wise constant strain, including torsion and shears, without any additional effort.

A. Piece-wise Constant Strain Kinematics

At any instant \( t \), considering the strain field \( \xi_n(X) \) constant along each of the \( N \) sections of the soft arm, we can replace the continuous field with a finite set of \( N \) twist vectors \( \xi_n \in \{1, 2, \ldots, N\} \), which play the role of the joint vectors of traditional rigid robotics. Under this assumption, equation (1) becomes an homogeneous, linear, matrix differential equation with constant coefficients, which can be analytically solved at any section \( n \) using the matrix
exponential method with the appropriate interval of X and initial value \([6]\). Going further into the details, the material abscissa \(X \in [0, L]\) is divided into \(N\) sections of the form \((0, L_1), (L_1, L_2) \ldots (L_{n-1}, L_n)\) (with \(L_N = L\)) and the initial value for the differential equation of the section \(n\) is given by the solution at the right boundary of the previous section \((X = L_{n-1})\). In other words, the solutions are glued together, on top of the other. With these considerations, the integration of (1) at a certain instant \(t\) becomes:

\[
g(X) = g(L_{n-1})e^{(X-L_{n-1})\hat{\xi}_n} .
\]

(5)

It turns out that the infinite series of the exponential in (5) can be expressed in a compact way as follows [22]:

\[
e^{(X-L_{n-1})\hat{\xi}_n} = I_4 + (X - L_{n-1})\hat{\xi}_n + \frac{1}{\theta_n^2} (1 - \cos ((X - L_{n-1}) \theta_n)) \hat{\xi}_n^2 + \frac{1}{\theta_n^3} ((X - L_{n-1}) \theta_n - \sin ((X - L_{n-1}) \theta_n)) \hat{\xi}_n^3 =: g_n(X)
\]

(6)

where \(\theta_n^2 = \frac{k_n^2}{\theta_n}\). For straight configurations of the section, it results \(\hat{\xi}_n^2 = 0\) and hence:

\[
e^{(X-L_{n-1})\hat{\xi}_n} = I_4 + (X - L_{n-1})\hat{\xi}_n
\]

circumventing the well known singularity for straight arm pose of the PCC models [23], [19]. Equation (6) can be viewed as the SE(3) counterpart of the Rodrigues formula for rotation (SO(3)). Calling \(g_n(X)\) the exponential function in (6), equation (5) can be written in the familiar way:

\[
g(X) = g(L_{n-1})g_n(X)
\]

(7)

which recursively returns the position and orientation of the micro-solid at \(X\) knowing the set of strains \(\hat{\xi}_n\) only.

Similarly, the velocity of each micro-solid \(\eta(X)\) can be obtained by a piece-wise integration of the continuum model (2). Under constant strains condition, at each section \(n\) and time \(t\), equation (2) is a non-homogeneous, linear, matrix differential equation with constant coefficients (remind that \(\hat{\xi}_n\) is piece-wise constant) which can be analytically solved using the variation of parameters method with the appropriate initial value [6].

\[
\eta(X) = e^{-(X-L_{n-1})\text{ad}_{\hat{\xi}_n}} \left( \eta(L_{n-1}) + \int_{L_{n-1}}^{X} e^{(x-L_{n-1})\text{ad}_{\hat{\xi}_n}} ds \right)
\]

(8)

Again, the exponential function in (8) can be expressed with a finite number of terms [22] (for the sake of presentation, \(x = X - L_{n-1}\) holds in the following).

\[
e^{\text{ad}_{\hat{\xi}_n}} = I_6 + \frac{1}{2\theta_n^2} (3 \sin(x \theta_n) - x \theta_n \cos(x \theta_n)) \text{ad}_{\hat{\xi}_n}^2 + \\
+ \frac{1}{2\theta_n^3} (4 - 4 \cos(x \theta_n) - x \theta_n \sin(x \theta_n)) \text{ad}_{\hat{\xi}_n}^3 + \\
+ \frac{1}{2\theta_n^4} (2 - 2 \cos(x \theta_n) - x \theta_n \sin(x \theta_n)) \text{ad}_{\hat{\xi}_n}^4 = \text{Ad}_{\hat{\xi}_n},
\]

(9)

where for straight configurations we have \(\text{ad}_{\hat{\xi}_n}^2 = 0\) and thus, taking the limit for \(\theta_n \to 0\), \(e^{\text{ad}_{\hat{\xi}_n}} = I_6 + \text{ad}_{\hat{\xi}_n}\). By inspecting the exponential (6), we can notice that this new exponential function is nothing else that the Adjoint representation (Ad) of the Lie group transformation \(g_n(X)\), defined as (together with its dual \(\text{Ad}^*\)):

\[
\text{Ad}_{\hat{\xi}_n} = \left( \begin{array}{cc} R_n & 0 \\ \bar{u}_n R_n & R_n \end{array} \right), \quad \text{Ad}^*_{\hat{\xi}_n} = \text{Ad}^{-T}_{\hat{\xi}_n} = \left( \begin{array}{cc} R_n & \bar{u}_n R_n \\ 0 & R_n \end{array} \right).
\]

With this definition at hand, equation (8) can be rewritten as follows:

\[
\eta(X) = \text{Ad}^{-1}_{\hat{\xi}_n(\eta)} \left( \eta(L_{n-1}) + \text{Ad}_{\hat{\xi}_n(X)} \hat{\xi}_n \right)
\]

(10)

where we have defined:

\[
\text{AD}_{\hat{\xi}_n}(X) := \int_{L_{n-1}}^{X} \text{Ad}_{\hat{\xi}_n(X)} ds = \\
xI_6 + \frac{1}{2\theta_n^2} (4 \cos(x \theta_n) - x \theta_n \sin(x \theta_n)) \text{ad}_{\hat{\xi}_n}^2 + \\
+ \frac{1}{2\theta_n^3} (2 - 2 \cos(x \theta_n) - x \theta_n \sin(x \theta_n)) \text{ad}_{\hat{\xi}_n}^3 + \\
+ \frac{1}{2\theta_n^4} (2x \theta_n - 3 \sin(x \theta_n) + x \theta_n \cos(x \theta_n)) \text{ad}_{\hat{\xi}_n}^4.
\]

(11)

Remarkably, equation (10) recursively compute the velocity of any micro-solid at \(X\) along the soft arm by virtue of the set of strains \(\hat{\xi}_n\) and strain rates \(\dot{\xi}_n\).

Finally, the acceleration of any micro-solid at \(X\) \(\dot{\eta}(X)\) can be calculated at any time \(t\) by means of a piece-wise integration of the continuous equation (3). Considering constant strains along one section, equation (3) is a non-homogeneous, linear, matrix differential equation with non-constant coefficients (given by the term \(\text{ad}_{\hat{\xi}_n}\) which is not constant wrt \(X\) due to \(\eta(X)\)). A direct application of the variation of parameters method with the appropriate initial value gives:

\[
\dot{\eta}(X) = e^{-\text{ad}_{\hat{\xi}_n}} \left( \dot{\eta}(L_{n-1}) + \int_{L_{n-1}}^{X} e^{\text{ad}_{\hat{\xi}_n}} (\dot{\xi}_n - \text{ad}_{\hat{\xi}_n} \eta) ds \right).
\]

(12)

Then, by virtue of the definitions of \(\text{Ad}_{\hat{\xi}_n}\), \(\text{Ad}^*_{\hat{\xi}_n}\) and the anticommutativity of the adjoint map, we obtain:

\[
\dot{\eta}(X) = \text{Ad}^{-1}_{\hat{\xi}_n(\dot{\eta})} \left( \dot{\eta}(L_{n-1}) + \int_{L_{n-1}}^{X} \text{Ad}_{\hat{\xi}_n(s)} \dot{\xi}_n ds \right).
\]

(13)

Let us focus on the last member inside the integral: \(\text{Ad}_{\hat{\xi}_n(s)} \dot{\xi}_n \hat{\xi}_n\). First, by means of equation (10) and the properties of the adjoint map, we can write:

\[
\text{Ad}_{\hat{\xi}_n(s)} \dot{\xi}_n \hat{\xi}_n = \text{ad}_{\dot{\eta}(L_{n-1})} \text{Ad}_{\hat{\xi}_n(s)} \hat{\xi}_n.
\]

Then, evoking the linearity and anticommutativity of the adjoint map, and using the equations (9) and (11), we obtain the equivalence:

\[
\text{Ad}_{\hat{\xi}_n(s)} \dot{\xi}_n \hat{\xi}_n = \text{ad}_{\dot{\eta}(L_{n-1})} \text{Ad}_{\hat{\xi}_n(s)} \hat{\xi}_n,
\]
which substituted in (13) gives the acceleration model as follows:

\[ \eta(X) = \text{Ad}_{\xi_n}^{-1} \left( \eta(L_{n-1}) + \text{Ad}_{\xi_n} \xi_n + \text{ad}_{\eta(L_{n-1})} \text{Ad}_{\xi_n} \xi_n \right). \]  

\( (14) \)

Again, equation (14) returns the acceleration of any micro-solid at \( X \) by means of the set of strains \( \xi_n \), strain rates \( \dot{\xi}_n \) and rates of strain rate \( \ddot{\xi}_n \) only.

The development above leads us to three kinematics equations (7), (10) and (14), which directly give a recursive model to calculate all the kinematic quantities from the knowledge of the joint space of the piece-wise soft arm, in a very similar fashion to traditional rigid robotics. Compared to the PCC model, the discrete Cosserat approach presented here is able to handle not only constant curvatures and elongations, but also shears and torsion, which are fundamental to deal with the strong interactions with the environment characteristic of locomotion and manipulation. Furthermore, the joint space composed by the \( N \) constant strains \( \xi_n \) is directly related to the configuration space \( g(X) \) through (7) (as well as to the velocity and acceleration through (10) and (14)), while the PCC model needs an additional map between the joint space and the arc parameters space, composed by the length, the curvature and the plane of bending of the section, before building the homogeneous transformation \( g(X) \).

**B. Single Section Constant Strain Dynamics**

In this section we obtain the generalized equation of motion of one section, leaving the general multi-section case for future research. Let us first specify the models of the distributed actuation, external load and internal forces appearing in (4) for the case of a soft robot arm moving in air. Considering the two most important actuation systems implemented in soft robotics, the cable driven and the fluidic actuation [21], we have respectively:

\[ \mathcal{F}_a(X,t) = - \left( \mathcal{F}_a^r + \text{ad}_{\xi_n}^* \mathcal{F}_a \right) \text{ and } \mathcal{F}_a(X,t) = 0, \]

where \( \mathcal{F}_a \) is the cable wrench acting on the micro-solid given by the cable tension and the cable path from the tip to the base [17], [20] (for which consistency is supposed to be constant along the section, i.e., \( \mathcal{F}_a(L_{n-1} < X < L_n) = \mathcal{F}_{a0} \) and \( \mathcal{F}_a^r = 0 \), Fig. 3). The model of the fluidic actuator, widely used in soft robotics nowadays [15], condensates the action of the pressure in a concentrated load at the tip of the section (Fig. 3), appearing in the boundary conditions.

Regarding the wrench of internal forces, a linear visco-elastic constitutive model, based on the Kelvin Voigt assumptions, is chosen [17].

\[ \mathcal{F}_i(X) = \Sigma (\xi_n - \xi_n^0) + \Upsilon \dot{\xi}_n =: \mathcal{F}_{i0}, \]

\( (15) \)

where \( \Sigma \) and \( \Upsilon \) are constant screw stiffness and viscosity matrices, equal to \( \Sigma = \text{diag}([G I EJ EA EA GA GA]) \), \( \Upsilon = \text{diag}([I 3J 3J 3J A A A A])\nu \), \( E \) being the Young modulus, \( G \) the shear modulus and \( \nu \) the shear viscosity modulus; \( \xi_n^0 = [0 0 0 1 0 0]^T \) is the reference straight configuration. No other assumptions behind the constitutive model are needed to describe the elastic behavior of the robot arm. Finally, as for the external loads, only the action of the gravity has been considered: \( \mathcal{F}_g = M \text{Ad}_{\xi_n}^{-1} G \), where \( G = [0 0 0 -9.81 0 0]^T \) is the gravity twist with respect to the inertial frame (in accordance with the choice of inertial frame given in figure 2).

In order to obtain the generalized equation of motion, we remind that the total power delivered by the distributed wrench \( \mathcal{F}(X) \) has to equate the power delivered by the generalized wrench \( \tau_1 \) or, in other words, the constrain forces do not generate any power [8]. This gives the following equivalence (for one section only):

\[ W = \int_0^L \mathcal{F}^T \eta dX = \int_0^L \mathcal{F}^T \text{Ad}^{-1}_{\xi_n} \text{Ad}_{\xi_n} \eta dX = \tau_1 \xi_n, \]

\( (16) \)

and consequently (since \( \text{Ad}^{-1}_{\xi_1} = \text{Ad}_{\xi_1}^* \)):

\[ \tau_1 = \int_0^L \text{Ad}_{\xi_n}^* \mathcal{F}(X) dX. \]

Thus, projecting both members of equation (4) with \( \text{Ad}_{\xi_n}^* \) and integrating over the interval \( [0,L] \), leads to the
following generalized dynamics equation for $\ddot{\xi}_1$.

$$
\int_0^L \left[ AD_{\xi_1}^T \left( F_1^T - F_{a} \right) + ad_{\xi_1}^* \left( F_1 - F_{a} \right) \right] dX
$$

$$
\ddot{\xi}_1 = \int_0^L AD_{\xi_1}^T g_{\xi_1} \left[ Ad^{*}_{\xi_1} \left( F_1 - F_{a} \right) \right] dX
$$

(17)

In order to calculate the rate of strain rate from (18), we first obtain the Adjoint map $Ad^{g}_{\xi_1}$ through (9) and his following generalized dynamics equation for $\ddot{\xi}_1$.

$$
\int_0^L AD_{\xi_1}^T g_{\xi_1} \left[ Ad^{*}_{\xi_1} \left( F_1 - F_{a} \right) \right] dX
$$

where properties of the adjoint map have been exploited in the last term. Now, making use of the identities $AD^{T}_{\xi_1} = Ad^{*}_{\xi_1}$ and $Ad^{*}_{\xi_1} \left( F_1^T + ad^{*}_{\xi_1} F \right) = \left( Ad_{\xi_1}^T F \right)^T$ and of an integration by parts and naming the coefficients matrices in squared parenthesis, we obtain the dynamic equation in the familiar way:

$$
M\ddot{\xi}_1 = \left( AD_{\xi_1}^T Ad^{*}_{\xi_1} \left( F_1 - F_{a} \right) \right) (L) - \int_0^L F_1 - F_{a} dX
$$

(18)

Remarkably, the integrals in (17) could be precalculated symbolically by a computer algebra system exploiting equations (9) and (11).

Let us now focus on the term involving the internal and actuation wrenches in (18), the more complex case of cable driven actuation has been considered, while the tip load case will be recovered at the end. In order to evaluate this quantity at the edges of the interval, we need to make use of the boundary conditions which, considering a constant cable actuation along the soft arm, state the following:

$$
F_i(L) = F_{a}(L) = F_{a_1}.
$$

Thus, we obtain the following expressions for cable driven actuation.

$$
\left( AD_{\xi_1}^T Ad^{*}_{\xi_1} \left( F_1 - F_{a} \right) \right) (L) = 0
$$

(19)

For what concerns fluidic actuation, we have $F_a = F_{a_1} = 0$ in (17) and $F_{a_1}$ appearing at the boundary. Hence we obtain:

$$
\left( AD_{\xi_1}^T Ad^{*}_{\xi_1} \left( F_1 \right) \right) (L) = AD_{\xi_1}^T (L) Ad^{*}_{\xi_1(L_1)} F_{a_1}
$$

$$
\int_0^L F_1 dX = L F_{a_1}
$$

(20)

Finally, ensured by the symmetric positive definiteness of the mass matrix $M$, we are able to solve equation (18) with respect to $\ddot{\xi}_1$, by means of the actuation input $F_{a_1}$ (both for cable driven and fluidic), the strains $\ddot{\xi}_1$ and the strain rates $\dot{\xi}_1$ only.

IV. SIMULATION RESULTS

In section III the piece-wise counterpart of the continuum kinematics equations (1), (2), (3) has been found as well as the generalized dynamics counterpart of one continuous section (4), which are respectively equations (7), (10), (14) and (18). In the following, simulation results are shown for both the multi-section kinematics and the single section dynamics.

A. Multi-Section Kinematics Simulation

Three sections have been considered of length 250 mm and circular cross section of radius 10 mm. At any time $t$, the set of strains $\xi_n$ and strain rates $\dot{\xi}_n$ are known. For instance, at $t = 0$ they are the initial conditions that for a straight configuration read $\dot{\xi}_1 = \dot{\xi}_2 = \dot{\xi}_3 = \dot{\dot{\xi}}_1 = \dot{\dot{\xi}}_2 = \dot{\dot{\xi}}_3 = [0 0 0 0 0 0]^T$. As rates of strain rate for each section $\ddot{\xi}_n(t)$, given as inputs, we chose periodic functions of 1 Hz frequency:

$$
\ddot{\xi}_1 = 150 \cos(2\pi t) [1 1 1 0 0 0]^T,
$$

$$
\ddot{\xi}_2 = -50 \sin(2\pi t) [1 1 1 0 0 0]^T,
$$

(21)

$$
\ddot{\xi}_3 = -150 \cos(2\pi t) [1 1 1 0 0 0]^T.
$$

As a first step, we calculate the set of homogeneous transformations $g_{n}(X)$ through equation (6), the corresponding Adjoint representations $Ad_{n}(X)$ through (9) and their definite integrals $AD_{n}(X)$ by means of (11). With these results, we are able to recursively calculate the homogeneous transformations $g(X)$ with (7), the screw velocities $\eta(X)$ thanks to (10) and the accelerations $\dot{\eta}(X)$ by means of (14). To do so we need to define:

$$
g(0) = \left( \begin{array}{cc} R(\pi) & 0 \\ 0 & 1 \end{array} \right), \eta(0) = 0, \dot{\eta}(0) = 0,
$$

representing respectively the relative position of the robot arm with respect to the environment (upside down as shown in figure 2), the velocity and the acceleration of the base (stationary in our case). Finally, a 4th order Runge-Kutta scheme allows us to update the set of $\xi_n$ and $\dot{\xi}_n$ for the next time step. Few snapshots of four seconds of simulation (four cycles) are shown in figure 4. As expected, a very simple time law for the rate of strain rates can produce quite complex shape configurations, which can be disclosed by the kinematics equations. As an example, the three components of the angular and linear velocities in each point $X$ of the last section are reported as a function of space $X$ and time $t$ (Fig. 5).

B. Single Section Dynamics Simulation

Only one section has been considered here, whose parameters are: $E = 110$ kPa, $\nu = 0$, $\mu = 10$ kPa-s, $L = 250$ mm and $r = 10$ mm. The Actuation inputs are constituted by a tip torque $M_a$ of magnitude $8$ mNm and a tip force $N_a$ of $40$ mN, both constant in time and aligned with the $z$ axis of the inertial frame. Thus, we converge to the case of tip actuation loads of (20), with $F_{a_n}$ equal to:

$$
F_{a_n} = \left( \begin{array}{cc} R(L)^T & 0 \\ 0 & L(R)^T \end{array} \right) (0,0,M_a,0,0,N_a)^T
$$

and $F_{a_1}$ given by (15). The rotation above projects the actuation loads from the inertial frame to the local frame at the tip of the section.

In order to calculate the rate of strain rate from (18), we first obtain the Adjoint map $Ad_{n}(X)$ through (9) and his
Fig. 4. Few snapshots of the multi-section kinematics simulation, respectively at $t = 0$, $t = 1.5$, $t = 2.5$ and $t = 4$. This shows how the kinematics model can handle torsional deformation for a multi-section robot arm.

definite integral with (11). With these results, we proceed in calculating the mass matrix $M$, the Coriolis matrix $C$ and the gravity matrix $G$, which allow us to solve the dynamics equation (18) for $\ddot{\xi}_n$. Finally, as in the kinematic case, a Runge-Kutta scheme allows us to update the set of $\ddot{\xi}_n$ and $\dot{\xi}_n$ for the next time step. For the seek of observation, during the process one can compute the kinematics quantities even if they are not needed in the single section algorithm.

Few snapshots of the dynamic simulation are shown in figure 6. The system, after some seconds, reaches a static equilibrium configuration which is completely out-of-the-plane and involve all the six components of the strain as shown in figure 7. Even though not explored before in the contest of piece-wise model, this situation is quite easy to appear in practice, as in the case of the FASTT robot soft legs [4] during contact with the ground.

V. FIRST EXPERIMENTAL COMPARISON

We validated the proposed model on a single BFA forming a leg of the FASTT robot. A BFA is mainly constituted by a hollow slender structure made by silicone rubber, externally reinforced by circumferential fibres to constraint the radial expansion and an inextensible fabric layer. When pressurized fluid is forced into the chamber, the presence of the reinforced bottom layer makes the extension of the longitudinal fibres non-uniform, producing the bending of the whole structure. We designed the chamber with a semi-circular geometry (Fig. 8), which has been shown to present lower bending resistance than full circle or rectangular sections [15]. The specimen material is the Smooth-on Ecoflex 00-30 silicone rubber (modulus at 100% $E_{100\%} = 69$ kPa, tensile strength 1.38 MPa), and its dimensions are: $R_2 = 6.4$ mm, $R_1 = 3.2$ mm, $h = 11.9$ mm, $L = 97.2$ mm (Fig. 8).

When actuated in free conditions, BFAs bend on the $(x-y)$ plane, perpendicular to the inextensible layer (Fig. 8). The neutral surface is forced to coincide with the surface of the inextensible layer ($y = 0$). As a consequence, we can compute the second area moment of the section respect to the $z$ axis and the position of the center of mass $c = (0, c_y, 0)^T$ as

$$c_y = \frac{\rho \int y dA}{\rho A}, J_z = \int y^2 dA.$$  \hspace{1cm} (22)

In turns, equations (22) are needed to build the local inertia tensor $J = \text{diag}(I_x, I_y, I_z)$, mass matrix $\mathcal{M} = \rho \left( \begin{array}{ccc} J & A \dot{c}^T & 0 \\ A & \frac{1}{2} A \dot{c}^T A & 0 \\ 0 & 0 & \frac{1}{2} A \dot{c}^T A \end{array} \right)$, stiffness matrix $\Sigma = \text{diag}(GI, EI_y, EI_z, EA + 100, GA + 100, GA)$ and viscosity matrix $\Upsilon = \text{diag}(I, 3I_x, 3I_z, 3A, A, A)\dot{\nu}$, where we have increased the linear elasticity to account for the inextensible fabric layer.

We conducted all the experiments with the actuator lying on a horizontal glass plate lubricated with oil to reduce the friction. We connected the internal chamber of the structure to a proportional electronic valve (Camozzi K8P), which controls the internal pressure of the actuator proportionally to the electronic signal received as input. We used a camera perpendicular to the bending plane to acquire the positions of the structure at 60 fps. From the videos, we tracked the $\varphi$ angle using the Kinovea 0.8.15 software.

A. Statics

The first validation has been conducted in quasi-static conditions. We obtained the bending angle $\varphi$ of the actuator for different values of the internal pressure $p$ (we refer to relative pressure). In a first model for BFAs, Polygerinos et al [15] reduced all the effects of the internal pressure $p$ to

\begin{align}
\text{Fig. 5. Component } x, y \text{ and } z \text{ of the angular and linear velocities for each point of the third section. Although we simulate a quite complex shape configuration, the results generated by the kinematics model could be used to achieve determined outcome.}
\end{align}
In order to test the ability of the model to capture the dynamics of the structure, we also conducted experiments to replicate the response of hyper-elastic materials for very high deformations. The inclusion of a more accurate material model will be considered for further works and lies outside the scope of this paper.

**B. Dynamics**

In order to test the ability of the model to capture the dynamics of the structure, we also conducted experiments...

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**Fig. 6.** Few snapshots of the single section dynamics simulation, from an oblique point of view (left) and a top point of view (right), respectively at $t = 0$ (top), $t = 1$ (middle) and $t = 5$ (bottom). This shows how the dynamics model can handle torsional deformations for a single section soft arm.

**Fig. 7.** Component $x$, $y$ and $z$ of the angular and linear strains which are all involved in a bending-pushing motion that can very easily appear in practice.

**Fig. 8.** Illustration of a single BFA in rest and deformed states. The angle $\phi$ between the tangents at the base and at the tip has been used to track the deformation in the experiments. Section AA shows the transversal section of the actuator.

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With the inclusion of both $M_t$ and $F_t$, we were able to successfully model the non-linear response of BFAs. We can observe that for high bending angles, the model and the experimental data start to diverge. This trend can be explained by the use of a linear material model, which fails in replicating the response of hyper-elastic materials for very high deformations. The inclusion of a more accurate material model will be considered for further works and lies outside the scope of this paper.

With the notation of the PCS model, the actuation and internal loads are simply obtained by defining: $\mathbf{F}_{a0} = (0,0,M_t,F_t,0,0)^T$, and $\mathbf{F}_{i0}$ given by equation (15). The experimental results and the comparison with the simulation of the theoretical model are shown in figure 9.

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$$M_t(p) = -p \int_0^\pi \int_0^{R_1} (r \sin \theta + h) r dr d\theta = -\frac{p}{6} (3\pi R_1^2 h + 4R_1^3).$$

(23)

From this equation, we can notice that $M_t$ depends linearly on $p$ but does not depend on the bending angle $\phi$. While the moment $M_t$ can be considered the main action that produces the bending when the actuator is at rest ($\phi = 0$), it fails to capture its behaviour when the deformation increases. In facts, from experimental results we obtained that these actuators show a highly non-linear response for $0 \leq \phi \leq \pi$ (a typical range of application) as shown in figure 9. The essential element lacking to reproduce this response is the pressure force at the tip $F_t = 0.5p\pi R_1^2$, which generates a moment $M_\phi$ respect to each section $X$. $M_\phi$, non-zero for $\phi > 0$ and depending on $\phi$, respect to a generic section $X$ equals (see Fig. 8)

$$M_\phi(p,\phi,X) = 2F_t(p)L \frac{\sin \left(\frac{\phi}{2} (1 - \frac{X}{L})\right)^2}{\phi(1-X/L)}. \tag{24}$$

Summarizing, the pressure at the tip of the actuator produces a force $F_t$, directed as the tangent at the tip and applied at the centroid of the transverse section of the chamber; however, the neutral surface is forced to coincide with the inextensible layer and so the transport of $F_t$ to the neutral surface produces also the moment $M_t$.

With the notation of the PCS model, the actuation and internal loads are simply obtained by defining: $\mathbf{F}_{a0} = (0,0,M_t,F_t,0,0)^T$, and $\mathbf{F}_{i0}$ given by equation (15). The experimental results and the comparison with the simulation of the theoretical model are shown in figure 9.
Fig. 9. Bending angle $\phi$ for different values of the applied pressure in quasi-statics conditions. Comparison between the theoretical and experimental results.

Fig. 10. Time evolution of the bending angle for two different frequencies: $f = 0.5$ Hz (top) and $f = 1.0$ Hz (bottom). Comparison between theoretical and experimental data.

with the pressure oscillating in time. In particular, we applied an internal pressure $p(t) = p_0 + \Delta \rho \cos(2\pi f t)$, with $p_0 = 0.18$ bar, $\Delta \rho = 0.08$ bar. The results of the time evolution of the bending angle for different frequencies $f$ are shown in figure 10.

VI. CONCLUSIONS

A new piece-wise constant strain model for soft robots dynamics has been presented based on a discrete Cosserat approach. With respect to the existing models, it allows torsion and shear deformations, which are fundamentals to cope with non-negligible external forces, without affecting the computational efficiency. Furthermore, the discrete Cosserat approach forms a unified framework with the continuum Cosserat approach for soft robots that can not be discretized in a finite number of sections, e.g., eel-like underwater robots [2]. A natural application of the present model is the terrestrial locomotion of soft crawlers, because of the out-of-the-plane external forces due to the contact with the ground. On this track, the model has been experimentally compared with a single BFA leg of the soft robot FASTT [4]. For future work, we will extend the dynamics equation (18) to the multi-section case along with an efficient recursive algorithm to solve it.

REFERENCES