A Nonparametric Riemannian Framework on Tensor Field with Application to Foreground Segmentation

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Overview

Motivation:
- Kernel density estimators (KDE) have been successful to model on Euclidean sample spaces the nonparametric nature of complex physical processes (e.g., time varying, non-static backgrounds).
- Nonparametrically reformulate the existing tensor-based GMM algorithms.
- The idea is to leave the data to show the underlying structure, instead of imposing one.

Issue:
- The tensor space (Symmetric Positive Definite matrices) is a Riemannian manifold.
- Applying a nonparametric approach outside Euclidean spaces is not trivial and requires careful use of differential geometry to deal with the Riemannian structure and curvature of the manifold.

Approach:
- Founded on the mathematically rigorous KDE paradigm on general Riemannian manifolds, we define a KDE specifically to operate on the tensor manifold.
- The tensor manifold is endowed with two Riemannian metrics: Affine-Invariant | Log-Euclidean.

Differential Geometry

A Riemannian manifold M is a topological space locally similar to an Euclidean space.
- A Riemannian manifold is a differentiable manifold M endowed with a Riemannian metric g.

\[ \frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}) \] 

basis of the tangent space, given a chart with a local coordinate system.

\[ g_{ij}(x) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_P \] 

the Riemannian metric is defined by a continuous collection of inner products on the tangent spaces.

\[ G_P(x) = [g_{ij}(x)]_P \] 

the metric can be expressed in that basis by a (n x n) symmetric, bilinear and positive-definite form called local representation of the Riemannian metric.

Kernel Density Estimation on Riemannian Manifolds

The idea is to build an analogue of a kernel on M by using a positive function of the geodesic distance on M, which is then normalized by the volume density function to take into account the curvature.
- The integral on a Riemannian manifold, depends on the point at which the kernel it is centered, e.g., depends on the local geometry of M in a neighborhood of the observation.

\[ K_{\kappa, K}(Z) = \frac{1}{N} \sum_i \frac{1}{h^n} K \left( \frac{D(Z, Z_i)}{h} \right) \] 

\[ \theta_P : Q \rightarrow \theta_P(Q) = \frac{\mu_{Q,E} + \mu_{Q,E}^T}{2} \left( \sqrt{G_P(y)} \right) \]

It is possible to ensure the integral is the same irrespective of where it is centered and make sure that the density function on M integrates to one by using the volume density function.
- If Q belongs to a normal neighborhood of P, then \( \theta_P(Q) \) is the density of the pullback of the volume measure on M to T_P M with respect to the Lebesgue measure on T_P M via the inverse exponential map at point P. It gives an indication of the curvature of the Riemannian space.

This is the same as the square-root of the determinant of the metric-tensor:

\[ \theta_P(Q) = \left( \sqrt{G_P(y)} \right) \]

The kernel estimator on Riemannian manifolds is consistent with standard kernel estimators on R and it converges at the same rate as the Euclidean kernel estimator.

Affine-Invariant Riemannian Metric

- An Affine-Invariant Riemannian metric can be deduced on the tensor manifold in terms of the Fisher information matrix.

\[ g_{ij} = g(E_i, E_j) = (E_i, E_j)_P = \frac{1}{4} (P^{-1}E_i P^{-1}E_j) \] 

Affine-Invariant metric for the tensor manifold derived from the Fisher matrix.

\[ d_Q(P, Q) = \sqrt{2 \log(\det(P^{-1}Q^{-1}))} \] 

Geodesic distance induced by the Affine-Invariant metric, derived from the Fisher matrix.

- Considering the normal coordinate system around P and the Ricci in this system and let y be the normal coordinates of Q:

\[ \{\theta_i\}_{i=1,...,n} = \{E_i\}_{i=1,...,n} \] 

\[ \sqrt{G_P(y)} = \left( 1 - \frac{y^T R y}{6} \right) \] 

\[ \forall i, j, k, l = 1,...,n \]

The Riemannian curvature for the tensor manifold, derived from the Fisher matrix, and the classical Levi-Civita affine connection:

\[ R_{ijkl} = R(E_i, E_j, E_k, E_l) = \frac{1}{4} \text{tr} \left( E_j P^{-1} E_k P^{-1} E_l P^{-1} \right) - \frac{1}{4} \text{tr} \left( E_j P^{-1} E_l P^{-1} E_k P^{-1} \right) \]

The Log-Euclidean Riemannian Metric

- The Log-Euclidean metric induces a space with a null curvature, while the theoretical properties are preserved.

\[ d_Q(P, Q) = \sqrt{\text{tr} \left( \log(P) - \log(Q) \right)^2} \] 

The geodesic distance induced by the Log-Euclidean metric, is extremely simplified.

- Endowed with the Log-Euclidean metric the tensor space is isomorphic, diffeomorphic and isometric to the associated Euclidean space of symmetric matrices.

- Endowed with the Log-Euclidean metric, the tensor manifold is a flat Riemannian space (sectional curvature is null everywhere).

When the Riemannian space is flat the volume density function is unity everywhere.

The isometry implies that the determinant of the metric tensor is unity everywhere.

\[ \left( \sqrt{G_P(y)} \right) = 1 \]

Experimental Results

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