A Nonparametric Riemannian Framework on Tensor Field with Application to Foreground Segmentation

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Abstract

Background modelling on tensor field has recently been proposed for foreground detection tasks. Taking into account the Riemannian structure of the tensor manifold, recent research has focused on developing parametric methods on the tensor domain e.g. gaussians mixtures (GMM) [7]. However, in some scenarios, simple parametric models do not accurately explain the physical processes. Kernel density estimators (KDE) have been successful to model, on Euclidean sample spaces the nonparametric nature of complex, time varying, and non-static backgrounds [8]. Founded on the mathematically rigorous KDE paradigm on general Riemannian manifolds [15], we define a KDE specifically to operate on the tensor manifold. We present a mathematically-sound framework for nonparametric modeling on tensor field to foreground segmentation. We endow the tensor manifold with two well-founded Riemannian metrics, i.e. Affine-Invariant and Log-Euclidean. Theoretical aspects are defined and the metrics are compared experimentally. Theoretic analysis and experimental results demonstrate the promise/effectiveness of the framework.

1. Introduction

Foreground detection is a crucial aspect in the understanding and analysis of video sequences. It is often described as the process that subdivides an image into regions of interest-object and background. This task usually relies on the extraction of suitable features that are highly discriminative. Statistical modeling in color/intensity space is a widely used approach for background modeling to foreground detection. However, there are situations where statistical modelling directly on image values isn't enough to achieve a good discrimination (e.g. dynamic scenes, illumination variation). Thus the image may be converted into a more information rich form, such as a *tensor field*, to yield latent discriminating features (e.g. color, gradients, filters responses). Texture is one of the most important features, therefore its consideration can greatly improve image analysis. The structure tensor [3, 10] has been introduced for such texture analysis as a fast local computation providing a measure of the presence of edges and their orientation.

For the sake of brevity, the related work description will be neither rigorous nor complete, but we want outline some of the key ideas (refer to [5] for a survey). Over the years, several background models have been proposed. These models can be broadly divided into *pixel-wise* and *block-wise* models. The *pixel-wise* models relie on the separation of statistical model for each pixel and the model is learned entirely from each pixel history. In the *block-wise* models, the pixel model depends not only on that pixel but also on the nearby pixels. They consider spatial information an essential element to understand the scene structure. Spatial variation information, such as gradient feature, helps improve the realiability of structure change detection. The pixel model also depends on its neighbors, taking advantage of the correlation existing between neighbouring pixels. Stauffer [21] proposed a parametrically approach in which each color pixel is represented as a GMM. The parameters are updated using an online Kmeans. However, in some scenarios, parametric models don't accurately explain the physical processes, i.e. can't model the nonparametric nature of complex, time varying, non-static backgrounds. One needs to employ nonparametric estimation techniques that don't make any assumptions about the *pdf*, except the mild assumption that *pdf* are smooth functions, and can represent arbitrary *pdfs* given sufficient data. Elgammal [8] proposed the KDE for modeling the pixel density, from its past samples. Foreground detection is performed by thresholding the probability of the observed sample.

The tensor space doesn't form a vector space, thus linear statistical techniques don't apply. Although the classical Euclidean operations are well adapted to general square matrices ($d \times d$), they are practically/theoretically unsatisfactory for tensors, which are very specific matrices, i.e. symmetric positive-definite ($\mathbf{S}_d^+$ is a Riemannian manifold : differentiable manifold equipped with a Riemannian metric).

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These problems have led to the use of Riemannian metrics. To circumvent these difficulties an Affine-Invariant metric [20, 11] has been proposed as a rigorous tensor framework. This metric has excellent theoretical properties and provide powerful processing tools, but essentially due to the curvature induced on the tensor space the computational burden can be high. To overcome this limitation, based on a novel vector space structure for tensors a new metric called Log-Euclidean was presented in [1]. A space with a null curvature is obtained, while the theoretical properties are preserved. In the Log-Euclidean framework, Riemannian computations become classical Euclidean computations in the matrix logarithms domain. This leads to simple extensions of the classical tools of vector statistics to tensors.

Background modelling on tensor field has become an important technique for foreground detection. In order to exploit all the tensor information and taking into account the Riemannian structure of the tensor manifold, previous work has focused on developing parametric methods. Caseiro *et al*. [6] proposed a foreground detection method for tensor-valued images based on the definition of GMM on the tensor domain. They proposed a online Kmeans to estimate the parameters based on the Affine-Invariant metric. In order to speed up the process, Caseiro *et al*. [7] presented a novel Kmeans algorithm based on the Log-Euclidean metric. They presented the theoretical aspects and the Affine-Invariant and Log-Euclidean frameworks are compared. From a practical point of view, results are similar, but the Log-Euclidean is much faster. However, as argued previously, sometimes the pixel data is more complex and can't be modeled parametrically. As shown by Elgammal [8] the KDE have been successful to model, on Euclidean sample spaces, the nonparametric nature of complex physical processes. Seeing that recently, in the mathematics community, was propose/defined the KDE on general Riemannian manifolds [15], it would be interesting to nonparametrically reformulate the existing tensor-based GMM algorithms [7, 6]. The idea is to leave the data to show the structure lying beyond them, instead of imposing one.

The main paper contributions are twofold: 1- Founded on the mathematically rigorous KDE on general Riemannian manifolds proposed in [15], we define a KDE specifically to operate on the tensor manifold. To accomplish this, the tensor manifold is endowed with two Riemannian metrics (Affine-Invariant,Log-Euclidean) and with a Euclidean metric to prove the benefits of take into account the Riemannian structure. **2-** We present a mathematically-sound framework for nonparametric modeling on tensor field to foreground detection. We generalized herein the nonparametric background model proposed in [8], one of the most widely used per-pixel models, from pixel domain to tensor domain. Our goal is to nonparametrically reformulate the tensor-based GMM proposed in [7] in a similar way to what Elgammal [8] did in relation to Stauffer's work [21]. Theoretical aspects are defined and the metrics are compared.

2. Riemannian Manifolds

A *manifold* is a topological space locally similar to an Euclidean space [14, 23, 4]. Let $\mathcal{M}$ be a n-manifold. For each point $\mathbf{P} \in \mathcal{M}$ there is a *coordinate chart* $(\mathcal{U}, \varphi)$ where $\mathcal{U}$ is a subset of $\mathcal{M}$ containing $\mathbf{P}$ and $\varphi\colon \mathcal{U} \rightarrow \tilde{\mathcal{U}}$ is a *homeomorphism* from $\mathcal{U}$ to a subset $\tilde{\mathcal{U}} = \varphi(\mathcal{U}) \subset \Re^n$. Given a chart $(\mathcal{U}, \varphi)$ the set $\mathcal{U}$ is called a *coordinate domain*. The map φ is denominated as *local coordinate map*, and the component functions of φ are called *local coordinates* on $\mathcal{U}$, i.e. the chart defines a *local coordinate system* $x = (x^1, ..., x^n)^T$. A Riemannian manifold is a differentiable manifold $\mathcal{M}$ endowed with a Riemannian metric g. The *tangent space* $T_{\mathbf{P}}\mathcal{M}$, defined $\forall \mathbf{P} \in \mathcal{M}$, is simply a vector space, attached to $\mathbf{P}$, which contains the *tangent vectors* to all curves on $\mathcal{M}$ passing through $\mathbf{P}$, i.e., the set of all tangent vectors at $\mathbf{P}$. A *Riemannian metric* is defined by a continuos collection of inner products $< .,. >_{\mathbf{P}}$ on the tangent space $T_{\mathbf{P}}\mathcal{M}$ for each $\mathbf{P} \in \mathcal{M}$. We denote this inner product by g and for two tangent vectors $u, v \in T_{\mathbf{P}}\mathcal{M}$, the inner product is written as $g_{\mathbf{P}}(u, v)$. The inner product induces a norm for u given by $||u|| = \sqrt{g_{\mathbf{P}}(u, u)}$. Given a chart $(\mathcal{U}, \varphi)$ at $\mathbf{P}$ with a local coordinate system $x = (x^1, ..., x^n)$, it is possible to determine a basis $\partial/\partial x = (\partial_1, ..., \partial_n)$ of the tangent space $T_{\mathbf{P}}\mathcal{M}$ (∂_i = shorter notation for $\partial/\partial x^i$). Any element of the $T_{\mathbf{P}}\mathcal{M}$ can be expressed in the form $\sum_{i=1}^{n} x^i \partial_i$. We can express the metric in this basis by a $(n \times n)$ symmetric, bilinear and positive-definite form $G_{\mathbf{P}}(x) = [g_{ij}(x)]_{\mathbf{P}}$ given by the inner products $g_{ij}(x) =< \partial_i, \partial_j >_{\mathbf{P}}$. The form $G_{\mathbf{P}}(x)$ is called the *local representation of the Riemannian metric*.

The *geodesic* is the locally length-minimizing smooth curve $\gamma(t)\colon I = [0, 1] \rightarrow \mathcal{M}$, characterized by the fact that it is *autoparallel*, e.g. the field of tangent vectors $\dot{\gamma}(t)$ stays parallel along γ (velocity is constant along the geodesic). In local coordinates, a curve is a geodesic *iff* it is the solution of the n second order Euler-Lagrange equations:

$$\frac{d^2 x^k}{dt^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \qquad \forall k = 1, ..., n \qquad (1)$$

where Γ_{ij}^k are the *Christoffel symbols* of the second kind. For each tangent vector $u \in T_{\mathbf{P}}\mathcal{M}$, there is a unique geodesic $\gamma(t)$ starting at $\mathbf{P}$ with initial velocity $\dot{\gamma}(0) = u$. The *exponential map*, $\exp_{\mathbf{P}}\colon T_{\mathbf{P}}\mathcal{M} \rightarrow \mathcal{M}$ maps $\dot{\gamma}(0) = u$ to the point reached by the geodesic, $\gamma(1) = \exp_{\mathbf{P}}(\dot{\gamma}(0))$. The origin of the $T_{\mathbf{P}}\mathcal{M}$ is mapped to the point itself, $\exp_{\mathbf{P}}(0) = \mathbf{P}$. For each $\mathbf{P} \in \mathcal{M}$, there exists a neighborhood $\tilde{\mathcal{U}}$ of the origin in $T_{\mathbf{P}}\mathcal{M}$, such that $\exp_{\mathbf{P}}$ is a *diffeomorphism* from $\tilde{\mathcal{U}}$ onto a neighborhood $\mathcal{U}$ of $\mathbf{P}$. In general, the exponential map is *onto* but only *one-to-one* in a neighborhood of $\mathbf{P}$. Over this neighborhood $\mathcal{U}$, we can define the inverse of the exponential map i.e. the mapping from $\mathcal{U}$ to $\tilde{\mathcal{U}}$ is the *logarithm map* $\log_{\mathbf{P}} = \exp_{\mathbf{P}}^{-1} : \mathcal{U} \to \tilde{\mathcal{U}}$. It maps any point $\mathbf{Q} \in \mathcal{U}$ to the unique tangent vector $u \in T_{\mathbf{P}}\mathcal{M}$ that is the initial velocity $\dot{\gamma}(0)$ of the geodesic between $\gamma(0) = \mathbf{P}$ and $\gamma(1) = \mathbf{Q}$. The neighborhood $\tilde{\mathcal{U}}$ is not necessarily convex. However, $\tilde{\mathcal{U}}$ is *star-shaped*, i.e. for any point $\in \tilde{\mathcal{U}}$, the line joining the point to the origin is contained in $\tilde{\mathcal{U}}$. The image of a star-shaped neighborhood under the exponential map is a neighborhood of $\mathbf{P}$ on the manifold (*normal neighborhood*). The exponential map can be used to define suitable coordinates for normal neighborhoods. Let $\tilde{\mathcal{U}}$ be a star-shaped neighborhood at the origin in $T_{\mathbf{P}}\mathcal{M}$ and let $\mathcal{U}$ be its image under the exponential map, i.e., $\mathcal{U}$ is a normal neighborhood of $\mathbf{P}$. Let e_i, $\forall i = 1, ..., n$ be a orthonormal coordinate system for $T_{\mathbf{P}}\mathcal{M}$. Therefore $g(e_i, e_j) = 0$ if $i \neq j$ and $g(e_i, e_j) = 1$ if $i = j$. The *normal coordinate system* of $\mathbf{P}$ is the coordinate chart $(\mathcal{U}, \varphi)$ which maps $\mathbf{Q} \in \mathcal{U}$ to the coordinates of $\log_{\mathbf{P}}(\mathbf{Q})$ in the orthonormal coordinate system $\Rightarrow \log_{\mathbf{P}}(\mathbf{Q}) = \sum_{i=1}^{n} \varphi^i(\mathbf{Q}) e_i$, where $\varphi^i(\mathbf{Q})$ is the i^{th} coordinate of $\varphi(\mathbf{Q}) \in \Re^n$.

Connection : The *curvature* concept play an important role in the expression of the KDE on manifolds. Before introducing the curvature notion, we need to precise the notion of *connection* ∇. It is crucial in geometry since it allows to transport quantities along curves in a consistent manner and, ultimately, to compare local geometries defined at different manifold locations [13]. The connection makes it possible to map any tangent space $T_{\mathbf{P}}\mathcal{M}$ onto another tangent space $T_{\mathbf{Q}}\mathcal{M}$. The need of such a mapping arises : imagine that we want to transport a vector, in a parallel manner, from its original point $\mathbf{P}$ to $\mathbf{Q}$. In general, the *parallel transport* procedure is dependent on the choice of coordinate system, which is not desirable. This dependence directly comes from the fact that the classical directional derivative does not behave well under changes of the coordinate system. It is possible to solve this problem, i.e. to make the differentiation intrinsic, by considering the *covariant derivative*. The covariant derivative is a way of specifying a derivative along tangent vectors of a manifold, i.e. orthogonal projection of the usual derivative of the vector fields onto tangent space. The canonical affine connection on a Riemannian manifold is the *Levi-Civita connection* [12] and is directly defined from the covariant derivative. It parallel transports a tangent vector along a curve while preserving its inner product (it is compatible with the metric, i.e. the covariant derivative of the metric is zero). The Levi-Civita coefficients are defined, in each local chart by the Christoffel symbols of the second kind Γ_{ij}^k as follows

$$\nabla_{ij}^k = \Gamma_{ij}^k = g^{kl}\Gamma_{ijl} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \tag{2}$$

$\forall i, j, k, l = 1, ..., n$, is used the Einstein's summation convention, g^{kl} is the metric inverse .

Riemannian Curvature Tensor (R) : The notion of curvature for Riemannian manifolds of dimension $\geqslant 3$ can't be fully described by a scalar quantity. Riemann introduced the Riemannian tensor (R), in terms of the Levi-Civita connection. It can be expressed in terms of the metric tensor and its first/second derivatives. The Riemann tensor measures the covariant derivatives non-commutativity. In local coordinates, is expressed through the Christoffel symbols as

$$\mathrm{R}_{ijk}^l = \partial_j \Gamma_{ki}^l - \partial_k \Gamma_{ji}^l + \Gamma_{jm}^l \Gamma_{ki}^m - \Gamma_{km}^l \Gamma_{ji}^m \tag{3}$$

Ricci Curvature Tensor ($\mathcal{R}$) : The Ricci is defined as the contraction of the Riemann curvature tensor (R) and can be thought of as the Laplacian of the Riemannian metric e.g. is a way to measure how much n-dimensional volumes in regions of an n-dimensional manifold differ from the volumes of equivalent regions in $\Re^n$. For Riemannian manifolds up to dimension 3 the Ricci completely describe its curvature. For manifolds of dimension $\geqslant 4$ it become insufficient. However, it plays an crucial role to define the KDE on the tensor space. The Ricci tensor is given as follows

$$\mathcal{R}_{ij} = \mathrm{R}_{ijk}^k = \mathrm{R}_{ijkl} g^{kl} \tag{4}$$

3. Background Modeling - Intrinsic

In this section we define a nonparametric estimator specifically to operate on the tensor manifold. The manifold is endowed with two Riemannian metrics and with the Euclidean metric to prove the benefits of take into account the Riemannian structure. The literature presents several methods to nonparametric modelling, e.g. Fourier expansions, splines, kernels [22]. We intend to use the kernel-based estimator known as *Parzen-window* estimator that essentially performs scattered-data interpolation by superposing kernel functions placed at each datum. It is the most widely-used practical method for nonparametrically estimate the underlying density of a random sample. By placing a smooth kernel, the resulting estimator will have a smooth density estimate. Sample spaces with a more complex intrinsic structure than the Euclidean space (e.g. Riemannian manifold structure) arise in a variety of contexts and motivate the adaptation of popular nonparametric estimation techniques on $\Re^n$. However, applying a nonparametric approach outside Euclidean spaces isn't trivial and requires careful use of differential geometry. Frequentist methods for nonparametric estimation on non-Euclidean spaces have been developed by Pelletier [15]. In [15], an appropriate kernel method is presented on general Riemannian manifolds, which generalizes the commonly used location-scale kernel on Euclidean spaces. The Pelletier's idea was to build an analogue of a kernel on $\mathcal{M}$ by using a positive function of the *geodesic distance* on $\mathcal{M}$, which is then normalized by the *volume density function* to consider the curvature. The Pelletier's estimator is consistent with standard kernel estimators on $\Re^n$. It also relies on the intuitive notion of a kernel function that has the highest value at the observation and monotonically-decreasing values with increasing distance from the observation. It converges at the same rate as the Euclidean kernel estimator. Provided the bandwidth is small enough, the kernel is centered on the observation, i.e. the observation is an intrinsic mean of its associated kernel. Consider a probability distribution with a density f on a Riemannian manifold $(\mathcal{M}, g)$. Let $\{\mathbf{Z}_1, ..., \mathbf{Z}_N\}$ be N i.i.d. random objects on $\mathcal{M}$ with density f. The density estimator of f is defined as the map $f_{N,K}\colon \mathcal{M} \to \Re_+$ which, to each $\mathbf{Z} \in \mathcal{M}$, associates the value $f_{N,K}(\mathbf{Z})$ given by

$$f_{N,K}(\mathbf{Z}) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\theta_{\mathbf{Z}_i}(\mathbf{Z})}\frac{1}{h^n}K\left(\frac{\mathrm{D}(\mathbf{Z},\mathbf{Z}_i)}{h}\right) \quad (5)$$

where $\mathrm{D}(\mathbf{Z}, \mathbf{Z}_i)$ is the geodesic distance between points $\mathbf{Z}, \mathbf{Z}_i \in \mathcal{M}$, $\theta_{\mathbf{Z}_i}(\mathbf{Z})$ is the volume density function, (n) is the manifold dimension, (h) is the bandwidth or smoothing parameter, (N) is the number of samples and $K(.)$ is a non-negative function (we define $K(.)$ as the Normal *pdf*).

In order to make sure that the density function on $\mathcal{M}$ integrates to one, we need to set up a framework that allows us to perform the integration. In a Euclidean space the integral of the kernel is independent of the point at which it is centered. For a Riemannian manifold, the integral depends on the point at which the kernel it is centered, e.g. depends on the local geometry of $\mathcal{M}$ in a neighborhood of the observation. This is necessary for obtaining an estimator which is consistent with kernel estimators on Euclidean space, and which possesses the same properties under a similar bunch of assumptions. It is possible to ensure the integral is the same irrespective of where it is centered by using the volume density function. This entails computing the ratio of the volume measures on the Riemannian manifold $\mathcal{M}$ and its tangent space $T_{\mathbf{P}}\mathcal{M}$ at each point. For $\mathbf{P}, \mathbf{Q} \in \mathcal{M}$, the volume density function $\theta_{\mathbf{P}}(\mathbf{Q})$ on $\mathcal{M}$ is given ([2], p.174)

$$\theta_{\mathbf{P}}\colon \mathbf{Q} \to \theta_{\mathbf{P}}(\mathbf{Q}) = \frac{\mu_{\exp_{\mathbf{P}}^* g}}{\mu_{g_{\mathbf{P}}}}\left(\exp_{\mathbf{P}}^{-1}\mathbf{Q}\right) \quad (6)$$

which is the quotient of the canonical measure of the Riemannian metric $\exp_{\mathbf{P}}^* g$ on $T_{\mathbf{P}}\mathcal{M}$ (pullback of the metric-tensor g by the exponential-map $\exp_{\mathbf{P}}$) by the Lebesgue measure of the Euclidean structure $g_{\mathbf{P}}$ on $T_{\mathbf{P}}\mathcal{M}$. In other words, if $\mathbf{Q}$ belongs to a normal neighborhood of $\mathbf{P}$, then $\theta_{\mathbf{P}}(\mathbf{Q})$ is the density of the pullback of the volume measure on $\mathcal{M}$ to $T_{\mathbf{P}}\mathcal{M}$ with respect to the Lebesgue measure on $T_{\mathbf{P}}\mathcal{M}$ via the inverse exponential-map $\exp_{\mathbf{P}}^{-1}$. It gives an indication of the curvature of the Riemannian space. This is the same as the *square-root of the determinant of the metric-tensor* expressed in the geodesic normal coordinates at $\mathbf{P}$ and evaluated at $\exp_{\mathbf{P}}^{-1}\mathbf{Q}$. Let $G_{\mathbf{P}} = [\, g_{ij} \,]_{\mathbf{P}}$, if $y = \varphi(\mathbf{Q}) = (y^1, ..., y^n)^T$ denotes the normal coordinates of $\mathbf{Q}$ in the normal coordinate system at $\mathbf{P}$ then $\theta_{\mathbf{P}}(\mathbf{Q}) = (\sqrt{|G_{\mathbf{P}}(y)|})$. In a normal neighborhood, θ is strictly positive and $\theta_{\mathbf{P}}(\mathbf{Q}) = \theta_{\mathbf{Q}}(\mathbf{P})$ [23][15].

3.1. Intrinsic : Euclidean Metric (E)

The distance D_e induced by the Euclidean metric is given by the Frobenius norm of the difference $\forall \mathbf{P}, \mathbf{Q} \in \mathbf{S}_d^+$ as [7]

$$\mathrm{D}_e(\mathbf{P}, \mathbf{Q}) = \sqrt{\mathrm{tr}((\mathbf{P}-\mathbf{Q})(\mathbf{P}-\mathbf{Q})^{\mathbf{T}})} \quad (7)$$

The Pelletier's estimator is consistent with KDE on Euclidean spaces, i.e. when $\mathcal{M}$ is the Euclidean space $\Re^n$, the estimator expression reduces to the one of a standard kernel estimator on $\Re^n$ [15]. Consider that $(\mathcal{M}, g)$ correspond to the Euclidean space $(\Re^n, \delta)$ where δ denotes the usual canonical Euclidean metric, and consider the canonical identification of the tangent space $T_{\mathbf{P}}\mathcal{M}$ at some point $\mathbf{P}$ of $(\Re^n, \delta)$, with $\Re^n$. Note that any two tangent spaces at different points are also canonically identified. This defines trivialy a normal chart, the domain of which is the entire manifold. In this chart, the components of the metric tensor form the identity matrix, hence $\forall \mathbf{P}, \mathbf{Q} \in \mathcal{M}$ the calculus of the $\theta_{\mathbf{P}}(\mathbf{Q})$ is $(\sqrt{|G_{\mathbf{P}}(y)|}) = 1$ (if the space is flat the volume density function is unity everywhere [2], p.154).

3.2. Intrinsic : Affine-Invariant Metric (AI)

Seeing that the manifold of the multivariate normal distributions with zero mean can be identified with the tensor manifold $\mathbf{S}_d^+$, a Riemannian metric on $\mathbf{S}_d^+$ can be deduced in terms of the Fisher information matrix [19]. An Affine-Invariant Riemannian metric [20, 11] for the tensor space $\mathbf{S}_d^+$, derived from the Fisher matrix, is given $\forall \mathbf{P} \in \mathbf{S}_d^+$ by

$$g_{ij} = g(E_i, E_j) = \langle E_i, E_j \rangle_{\mathbf{P}} = \frac{1}{2}\mathrm{tr}(\mathbf{P}^{-1}E_i\mathbf{P}^{-1}E_j) \quad (8)$$

$\forall i, j = (1, ..., n)$; $\{\partial_i\}_{i=1,...,n} = \{E_i\}_{i=1,...,n}$ denote the canonical basis of the tangent space of $\mathbf{S}_d^+$; The formulae of the exponential/logarithm maps, for $\mathbf{S}_d^+$ endowed with this metric can be found in [7]. The geodesic distance D_a induced by the Affine-Invariant metric, derived from the Fisher information matrix, $\forall \mathbf{P}, \mathbf{Q} \in \mathbf{S}_d^+$ [7] is given by

$$\mathrm{D}_a(\mathbf{P}, \mathbf{Q}) = \sqrt{\frac{1}{2}\mathrm{tr}(\log^2(\mathbf{P}^{-\frac{1}{2}}\mathbf{Q}\mathbf{P}^{-\frac{1}{2}}))} \quad (9)$$

Square-Root Determinant Metric $(\sqrt{|G_{\mathbf{P}}(y)|})$: Generalizing the *pdf* concept requires a measure $d\mathcal{M}$ on the manifold which, in case of Riemannian manifolds, is induced in a natural way by the metric $G(x)$ for a given local coordinate system [16]. As any metric in a Euclidean space, the Riemannian metric induces an infinitesimal volume element $d\mathcal{M}(x) = \sqrt{|G(x)|}dx$ in any chart (volume of the parallelepiped spanned by the vectors of an orthonormal basis of the tangent space). The difference is that the measure is now different at each point since the local expression of the metric is changing. The *reference measure* $d\mathcal{M}(x)$ on the manifold can be used to measure *random events* on the manifold (generalization of random variables), and to define their *pdf* (if it exists), i.e. the function $p(x)$ on the manifold such that the respective *probability measure* is given by $dP(x) = p(x)d\mathcal{M}(x)$. The induced measure $d\mathcal{M}$ actually represents the notion of *uniformity* according to the chosen Riemannian metric. With the probability measure dP of a random element, we can integrate functions $\phi(x)$ from the manifold to any vector space, thus defining the expected value of this function. This notion of expectation corresponds to the one we defined on real random variables and vectors. Seeing that the Taylor expansion of the metric is defined in [23], Pennec [16] deduced a Taylor expansion of the measure $d\mathcal{M}$ in a normal coordinate system around the mean value, in order to generalized a Normal law to Riemannian manifolds. In our case we consider the normal coordinate system around $\mathbf{P} \in \mathcal{M}$. The Taylor expansion of the measure $d\mathcal{M}$ around the origin is given as [16]

$$d\mathcal{M}_{\mathbf{P}}(y) = \left(\sqrt{|G_{\mathbf{P}}(y)|}\right) dy \approx \left(1 - \frac{y^T \ \mathcal{R} \ y}{6}\right) dy \quad (10)$$

where y is the normal coordinates of $\mathbf{Q} \in \mathcal{M}$ and $\mathcal{R}$ is the Ricci tensor in the considered normal coordinate system. To define the Ricci tensor for $\mathbf{S}_d^+$, we have to choose an affine connection, since this will influence the curvature properties. The existence/uniqueness of the Riemannian barycenter, requires that the space exhibit a non-positive sectional curvature. The canonical affine connection on a Riemannian manifold is known as the *Levi-Civita connection* (or covariant derivative). It is the only one to be compatible with the metric (covariant derivative of the metric is zero), i.e., the only one by which the parallel transport of a vector does not affect its length. Therefore, we will work with the Levi-Civita connection in the remaining developments. Using the local coordinates, the Christoffel symbols of the second kind [20] for the space $\mathbf{S}_d^+$ can also be expressed in terms of the elements of the canonical $\{E_i\}_{i=1,\dots,n}$ and dual basis $\{E_i^*\}_{i=1,\dots,n}$ (of the cotangent space of $\mathbf{S}_d^+$)

$$\Gamma_{ij}^k = \Gamma(E_i, E_j; E_k^*) = E_k^*.(\nabla_{E_i}^F E_j) \quad (11)$$

$\forall i, j, k = 1, ..., n$. Provided that [20] ($\mathbf{P} \in \mathbf{S}_d^+$) :

$$\begin{aligned} \frac{\partial g(E_i, E_j)}{\partial x_k} &= -\frac{1}{2}\text{tr}\left(\mathbf{P}^{-1}E_k\mathbf{P}^{-1}E_i\mathbf{P}^{-1}E_j\right) \\ &\quad -\frac{1}{2}\text{tr}\left(\mathbf{P}^{-1}E_i\mathbf{P}^{-1}E_k\mathbf{P}^{-1}E_j\right) \end{aligned} \quad (12)$$

the unique affine connection (Levi-Civita) associated with the Fisher information metric was derived from Eq. 2 as

$$\Gamma(E_i, E_j; E_k^*) = -\frac{1}{2}\text{tr}\left(E_i\mathbf{P}^{-1}E_jE_k^*\right) - \frac{1}{2}\text{tr}\left(E_j\mathbf{P}^{-1}E_iE_k^*\right) \quad (13)$$

Riemannian Curvature Tensor (R) : The Riemann tensor for $\mathbf{S}_d^+$, derived from the Fisher information metric, and the classical Levi-Civita affine connection is given by [20]

$$\begin{aligned} \mathrm{R}_{ijkl} &= \mathrm{R}(E_i, E_j, E_k, E_l) \\ &= \frac{1}{4}\text{tr}\left(E_j\mathbf{P}^{-1}E_i\mathbf{P}^{-1}E_k\mathbf{P}^{-1}E_l\mathbf{P}^{-1}\right) \\ &\quad -\frac{1}{4}\text{tr}\left(E_i\mathbf{P}^{-1}E_j\mathbf{P}^{-1}E_k\mathbf{P}^{-1}E_l\mathbf{P}^{-1}\right) \end{aligned} \quad (14)$$

Ricci Curvature Tensor $(\mathcal{R})$: The Ricci is computed on the basis of closed-form expressions for the metric and R (Eq. 4) and simply involves traces of matrix products. Symbolic computations easily lead to the components of the Ricci in terms of the components of $\mathbf{P}^{-1}$. Comparing the Ricci with the metric, we confirm that the $\mathbf{S}_d^+$ endowed with this metric isn't an *Einstein* manifold [14] i.e. it's a space of non-constant non-positive curvature, for which there doesn't exist a constant L such that $\mathcal{R}_{ij} = Lg_{ij}$. We need take into account R to deal with the curvature.

3.3. Intrinsic : Log-Euclidean Metric (LE)

The geodesic distance D_l induced by the Log-Euclidean metric, $\forall \mathbf{P}, \mathbf{Q} \in \mathbf{S}_d^+$ is extremely simplified as follows [1]

$$\mathrm{D}_l(\mathbf{P}, \mathbf{Q}) = \sqrt{\text{tr}\left((\log(\mathbf{Q}) - \log(\mathbf{P}))^2\right)} \quad (15)$$

By trying to put a Lie group structure on $\mathbf{S}_d^+$, Arsigny [1] observed that the matrix exponential is a *diffeomorphism* from the Euclidean space of symmetric matrices $\mathbf{S}_d$ to the tensor space $\mathbf{S}_d^+$. The important point here is that the logarithm of a tensor $\mathbf{P} \in \mathbf{S}_d^+$ is unique, well defined and is a symmetric matrix $u = \log(\mathbf{P})$. The matrix exponential of any symmetric matrix u yields a tensor $\mathbf{P} = \exp(u)$. Thus, one can seamlessly to transport all the operations defined in the vector space of $\mathbf{S}_d$ to the $\mathbf{S}_d^+$. Since there is a one-to-one mapping between $\mathbf{S}_d^+$ and $\mathbf{S}_d$, one can transfer to tensors the standard algebraic operations (addition $+$ and scalar multiplication .) with the matrix exponential. This defines on $\mathbf{S}_d^+$ the *logarithmic multiplication* $\odot$ and the *logarithmic scalar multiplication* $\circledast$ operators, which provides $\mathbf{S}_d^+$ with a commutative *Lie group structure* and with a *Vector space structure* [1]. The $\odot$ gives a commutative Lie group structure to $\mathbf{S}_d^+$, for which any metric at the tangent space at the *identity* is extended into a *bi-invariant* Riemannian metric on $\mathbf{S}_d^+$ (invariant by multiplication and inversion) e.g. the Euclidean metric on $\mathbf{S}_d$ is transformed into a bi-invariant Riemannian metric on $\mathbf{S}_d^+$. Among Riemannian metrics in Lie groups, the most suitable in practice, when they exist, are bi-invariant metrics. They are used to generalize to Lie groups a notion of mean consistent with multiplication and inversion. For our tensor Lie group, bi-invariant metrics exist and are simple. Their existence results from the commutativity of logarithmic multiplication between tensors, and since correspond to Euclidean metrics in the logarithms domain are called *Log-Euclidean* metrics. With $\circledast$, we get a complete structure of vector space on tensors, meaning that most of the operations that were generalized using minimizations for the Affine-Invariant do have a closed-form with a Log-Euclidean. The Riemannian framework is extremely simplified. Results obtained on logarithms are mapped back to the tensor domain with the exponential [1].

The Lie group of $\mathbf{S}_d^+$ is *isomorphic* (algebraic structure of vector space is conserved) and *diffeomorphic* to the additive group of $\mathbf{S}_d$. The Lie group of $\mathbf{S}_d^+$ endowed with a Log-Euclidean metric, is *isometric* (distances are conserved) to $\mathbf{S}_d$ endowed with the associated Euclidean metric. The Log-Euclidean metric induces on $\mathbf{S}_d^+$ a space with a null curvature, i.e. endowed with the Log-Euclidean, the $\mathbf{S}_d^+$ is a flat Riemannian space (sectional curvature ([9], p.107) is null everywhere). As proved in ([2], p.154), when the Riemannian space is flat the volume density function is unity everywhere. Taking into account these facts, and considering that the *isometry* implies that the determinant of the metric tensor is unity everywhere [4], the calculus $\forall \mathbf{P}, \mathbf{Q} \in \mathbf{S}_d^+$ of $\theta_{\mathbf{P}}(\mathbf{Q})$ is extremely simplified to $(\sqrt{|G_{\mathbf{P}}(y)|}) = 1$.

4. Background Modeling - Extrinsic

The KDE was *intrinsically* formulated to operate on $\mathbf{S}_d^+$. Depending on the metric choosed the estimation using this formulation can be hard to carry out. It would be interesting to *extrinsically* reformulate the KDE and evaluate its efficiency. We propose an extrinsic KDE designed to operate on $\mathbf{S}_d^+$ with two Riemannian metrics. The extension is extrinsic in the sense that the inherent estimation is performed on the tangent spaces. By first mapping the data to a tangent space, that is a vector space, we can use a Euclidean KDE [8]. We start by defining mappings from neighborhoods on the manifold to a euclidean space, similar to coordinate charts. Our maps are the logarithm maps $\log_{\mathbf{P}}$ that map the neighborhood of points $\mathbf{P} \in \mathcal{M}$ to the tangent space $T_{\mathbf{P}}\mathcal{M}$. Since this mapping is a homeomorphism around the neighborhood of a point, the manifold structure is locally preserved. This requires choosing a suitable tangent space on which to map. In this work, the data was mapped onto the tangent space at the mean point. Since the Karcher mean $\mu \in \mathcal{M}$ is the minimizer of the sum of squared Riemannian distances [17], and the mapping preserves the structure of the manifold locally, this tangent space is a good choice. This procedure can be seen as a way of linearizing the manifold around μ since the $T_\mu\mathcal{M}$ provides a first order approximation of the manifold around μ. This is the same to consider a normal coordinate system $(\mathcal{U}, \varphi)$ around μ. At time t, let $\{\mathbf{Z}_i\}_{i=1,\ldots,N}$ be the set of N points on $\mathcal{M}$ (past samples/observations) and $\mathbf{Z}_0 \in \mathcal{M}$ is the actual sample that we want to classify. First, we compute the mean $\mu_t \in \mathcal{M}$ of all samples $\{\mathbf{Z}_i\}_{i=0,\ldots,N}$. Then, we map (project) all points $\{\mathbf{Z}_i\}_{i=0,\ldots,N}$ to the tangent space $T_{\mu_t}\mathcal{M}$, using the logarithm map $\log_{\mu_t}(\mathbf{Z}_i)$, $i = 0, ..., N$. Let $z_i = \varphi(\mathbf{Z_i}) = (z_i^1, ..., z_i^n)^T$ denote the normal coordinates of $\mathbf{Z_i}$, $\forall i = 0, ..., N$ in the normal coordinate system at μ_t. Seeing that the normal system defines a vector space, we can apply the Euclidean KDE on $\Re^n$ [8].

4.1. Extrinsic : Affine-Invariant Metric (AI)

Using the AI metric a closed-form expression for the mean on $\mathbf{S}_d^+$ cannot be obtained. But a gradient descent algorithm was presented in [11]. The mean is only *implicitly* defined since the Riemannian barycenter exists and is unique for nonpositive sectional curvature manifolds [20]. The algorithm alternates the barycenter computation in the exponential chart centered at the current estimation of the mean value, and a re-centering step of the chart at the point of the manifold that corresponds to the computed barycenter. An exact implementation of this iterative algorithm can be a costly procedure. In order to speed up the process, we will use a method based on a online Kmeans on $\mathbf{S}_d^+$ (endowed with the Affine-Invariant metric) proposed by Caseiro *et al*. [6]. At each frame, the mean value $\mu_t \in \mathbf{S}_d^+$ is updated using a learning rate (ρ). The new mean μ_t combine the prior information $\mu_{t-1} \in \mathbf{S}_d^+$ with the actual sample $\mathbf{Z}_0 \in \mathbf{S}_d^+$. To take into account the manifold geometry, Caseiro *et al*. [6] derived an approximation equation to update the tensor mean, based on the concept of interpolation. The interpolation can be seen as a walk along the geodesic joining the tensors. After some mathematical simplifications [6] the mean update equation turns into

$$\mu_t = (\mu_{t-1})^{\frac{1-\rho}{2}} \ (\mathbf{Z}_0)^{\rho} \ (\mu_{t-1})^{\frac{1-\rho}{2}} \tag{16}$$

4.2. Extrinsic : Log-Euclidean Metric (LE)

A special case using the Affine-Invariant metric is given by mapping to the identity. The identity matrix $\in \mathbf{S}_d^+$ and so mapping to its tangent plane instead of the mean tensor is valid. This is equivalent to using the Log-Euclidean metric. The Log-Euclidean framework defines a mapping where the tensor space $\mathbf{S}_d^+$ is isomorphic, diffeomorphic, and isometric to the associated Euclidean space of symmetric matrices $\mathbf{S}_d$. This mapping is precisely the simple matrix logarithm ($\log_{\mathbf{I}} \mathbf{P} = \log \mathbf{P}$), $\forall \mathbf{P} \in \mathbf{S}_d^+$ i.e. tensors are transformed into symmetric matrices using $\log \mathbf{P}$. Since the Log-Euclidean transforms Riemannian computations on tensors into Euclidean computations on vectors in the logarithms domain, practically one simply uses the usual tools of Euclidean statistics on the logarithms and maps the results back to the tensor vector space with the exponential. Notice, that in practice this extrinsic KDE algorithm in which the mapping is defined by the matrix logarithm $\log$ is mathematically equivalent to the intrinsic KDE using the Log-Euclidean metric presented in the Section 3.3.

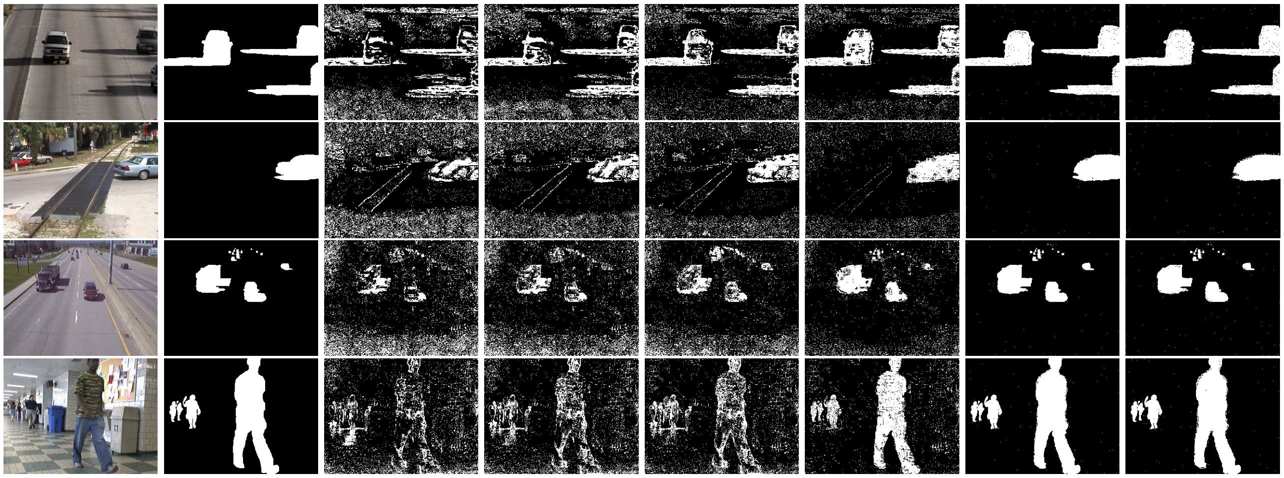

Figure 1. Examples of segmentation results on the sequences 1 to 4 (**Top** $\mapsto$ **Bottom**) using the methods : GroundTruth ; GMM[**I,Ix,Iy**] ; KDE[**I,Ix,Iy**] ; KDE-Int[**Tensor**]-**E** ; GMM[**Tensor**]-**AI** ; KDE-Int[**Tensor**]-**AI** ; KDE-Int[**Tensor**]-**LE** (**Left** $\mapsto$ **Right**)

5. Results

In order to analyze/confirm the effectiveness of the proposed framework, we conduct experiments on six challenging video sequences (including the two sequences used in [7]) presented in previous literature, including indoor/outdoor environments with some complex backgrounds (e.g. illumination changes, small camera motions). The sequence 1 (**HighWayI**) is a highway scenario where the vast majority of car colors are shades of gray (similar to the background). The sequence 2 (**Railway**) is the moving camera sequence used by Caseiro *et al*. in [7], which involved a camera mounted on a tall tripod. The wind caused the tripod to sway back and forth causing motion of the camera. The sequence 3 (**HighWayIII**) is a highway scenario where there is typically a steady stream of vehicles. The sequence 4 (**HalwayI**) shows a busy hallway where people are walking or standing still. The sequence 5 (**Campus**) is a noisy sequence from outdoor campus site where cars approach to an entrance barrier and students are walking around. The sequence 6 (**HighWayII**) is a highway scenario where the camera presents some motion and the image is noisy. The sequences 1,5,6 are from the ATON project (http://cvrr.ucsd.edu/aton/shadow) or VISOR repository (http://www.openvisor.org/) [18]. The sequence 2 is from Yaser Seikh work (http://www.cs.cmu.edu/∼yaser/). The sequences 3,4 are from Nicolas Brisson work (http://cvrr.ucsd.edu/aton/shadow/). Two widely-used vector space methods to foreground detection, namely the Stauffer's [21] (GMM) and the Elgammal's [8] (KDE) techniques are employed to compare with the proposed tensor framework (using two types of features sets, i.e. color [**r, g, b**] and gray level incremented with gradients [**I, Ix, Iy**]). Our framework (KDE[**Tensor**]) is also compared with the recently proposed parametric technique to background modeling on tensor field [7] (GMM[**Tensor**]). Both the GMM and KDE tensor frameworks are tested using two Riemannian metrics, i.e. Affine-Invariant (**AI**) and Log-Euclidean (**LE**) and with the standard Euclidean metric (**E**) to comprove the benefits of take into account the Riemannian structure of the manifold. We also compared the intrinsic KDE tensor (KDE-Int[**Tensor**]) formulation (section 3) with the extrinsic (KDE-Ext[**Tensor**]) counterpart (section 4). The performance comparison of the methods is based primarily on a quantitative evaluation in terms of true positive ratio (**TPR**) and false positive ratio (**FPR**). The comparative results of these experiments are presented in Table 1. Examples of segmentation results on four sequences are shown in Figure 1. Note that the results presented here are raw data without any postprocessing. In our work we used a structure tensor in which are encoded the gray information [**I**] and texture [**Ix, Iy**] features (gradients). This result in a structure tensor defined as $\mathbf{T} = (vv^T)$ with $v =$ [**I, Ix, Iy**] and $\mathbf{T} \in \mathbf{S}_3^+$ (d=3 and n=6). The structure tensor is calculated for each image pixel using a patch with dimension 3×3. All the tensor-based experiments presented (KDE[**Tensor**], GMM[**Tensor**]) use the same structure tensor. Remark that the advantage of the nonparametric paradigm over the parametric counterpart remains independent of the information included in the structure tensor, i.e. the proof of concept does not change. Due to space limitations, we do not discuss implementation technicalities. We recall that the paper ideia is to generalize the nonparametric background model proposed by Elgammal *et al*. [8], from pixel domain to tensor domain. Therefore, using the KDE derivations for the tensor manifold, the algorithm to foreground detection is basically similar to [8]. The Log-Euclidean metric considerably simplifies the scheme, from a practical point of view, while maintaining the mathematical soundness. Notice that the (KDE[**Tensor**] + LogEuclidean) is faster than all the (GMM[**Tensor**]) versions.

	1-HighWayI		2-Railway		3-HighWayIII		4-HalwayI		5-Campus		6-HighWayII	
Methods	TPR	FPR	TPR	FPR	TPR	FPR	TPR	FPR	TPR	FPR	TPR	FPR
GMM [**r, g, b**]	52.10	20.25	55.23	23.10	55.40	24.95	50.10	25.30	48.50	34.48	50.63	37.14
GMM [**I, Ix, Iy**]	58.05	17.58	60.45	20.80	61.05	22.35	55.05	21.90	54.30	31.05	55.12	34.03
KDE [**r, g, b**]	59.83	16.85	62.70	19.95	63.15	22.00	57.45	21.40	55.60	30.13	58.25	32.58
KDE [**I, Ix, Iy**]	65.95	15.10	68.55	16.90	69.03	20.85	64.73	18.85	60.10	27.65	63.90	28.10
GMM [**Tensor**] - **E**	68.02	14.00	72.90	14.20	70.24	17.05	67.95	15.18	64.21	24.59	65.04	25.04
GMM [**Tensor**] - **AI**	83.90	07.95	83.25	07.38	81.70	09.35	82.27	10.25	74.94	14.01	73.81	15.20
GMM [**Tensor**] - **LE**	83.00	08.21	82.10	07.92	80.96	09.94	82.93	10.96	72.82	14.93	73.02	15.86
KDE-Int [**Tensor**] - **E**	74.10	10.36	76.30	10.65	74.35	10.46	73.05	11.03	69.04	15.08	70.13	18.83
KDE-Int [**Tensor**] - **AI**	96.25	01.02	94.35	01.74	95.65	00.95	95.78	01.12	87.90	05.52	87.25	07.95
KDE-Int [**Tensor**] - **LE**	95.64	01.17	94.23	01.96	94.75	01.08	95.53	01.95	87.14	05.71	86.97	07.13
KDE-Ext [**Tensor**] - **AI**	90.05	04.10	89.45	04.51	89.95	05.01	90.03	05.93	81.42	10.76	79.53	12.07
KDE-Ext [**Tensor**] - **LE**	95.64	01.17	94.23	01.96	94.75	01.08	95.53	01.95	87.14	05.71	86.97	07.13

Table 1. Quantitative performance evaluation on the six sequences, in terms of : True positive ratio (**TPR**) and False positive ratio (**FPR**)

6. Conclusion

Taking into account the Riemannian structure of the tensor manifold, we derived a nonparametric Riemannian framework for foreground detection on tensor field. We presented the necessary background about differential geometry, i.e. we focus on the main geometric concepts of Riemannian manifolds, nonparametric estimation on such manifolds and the respective extensions to the tensor manifold endowed with two Riemannian metrics. The explicit formulation of a KDE on tensor manifold respecting the non-flat nature of the space as well as the respective application to the foreground detection problem are the core contributions of the paper. Overall we conclude that the consequent usage of the intrinsic Riemannian structure of the manifold for model derivation in conjunction with a suitable nonparametric estimation scheme for the underlying density, yields the most accurate and reliable approach to foreground detection from tensor-valued images presented so far.

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