A Nonparametric Riemannian Framework on Tensor Field with Application to Foreground Segmentation

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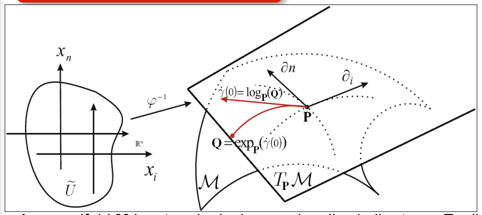
Overview

- **Motivation** :
  - Kernel density estimators (**KDE**) have been successful to model on Euclidean sample spaces the nonparametric nature of complex physical processes (*e.g.* time varying , non-static backgrounds) .
  - Nonparametrically reformulate the existing tensor-based **GMM** algorithms.
  - The idea is to leave the data to show the underlying structure, instead of imposing one.
- **Issue** :
  - The tensor space (**Symetric Positive Definite** matrices) is a Riemannian manifold.
  - Applying a nonparametric approach outside Euclidean spaces is not trivial and requires careful use of differential geometry to deal with the Riemannian structure and curvature of the manifold.
- **Approach** :
  - Founded on the mathematically rigorous KDE paradigm on general Riemannian manifolds we define a **KDE specifically to operate on the tensor manifold**.
  - The tensor manifold is endowed with two Riemannian metrics : **Affine-Invariant** | **Log-Euclidean**

Differential Geometry



Levi-Civita Connection

$$\nabla^k_{ij} = \Gamma^k_{ij} = g^{kl}\Gamma_{ijl} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

Riemannian Curvature Tensor

$$\mathrm{R}^l_{ijk} = \partial_j\Gamma^l_{ki} - \partial_k\Gamma^l_{ji} + \Gamma^l_{jm}\Gamma^m_{ki} - \Gamma^l_{km}\Gamma^m_{ji}$$

Ricci Curvature Tensor

$$\mathcal{R}_{ij} = \mathrm{R}^k_{ijk} = \mathrm{R}_{ijkl}g^{kl}$$

- A *n*-manifold **M** is a topological space locally similar to an Euclidean space.
- A **Riemannian manifold** is a differentiable manifold **M** endowed with a Riemannian metric *g*.

$\forall i,j,k,l = 1,\dots,n$

$\partial/\partial x = (\partial_1, \dots, \partial_n)$ ➡ basis of the tangent space, given a chart with a local coordinate system.

$g_{ij}(x) = <\partial_i, \partial_j>_{\mathbf{P}}$ ➡ the Riemannian metric is defined by a continuos collection of inner products on the tangent spaces.

$G_{\mathbf{P}}(x) = [g_{ij}(x)]_{\mathbf{P}}$ ➡ the metric can be expressed in that basis by a (n x n) symmetric, bilinear and positive-definite form called **local representation of the Riemannian metric**.

Kernel Density Estimation on Riemannian Manifolds

- The idea is to build an analogue of a kernel on **M** by using a positive function of the **geodesic distance** on **M**, which is then normalized by the **volume density function** to take into account the curvature.
- The integral on a Riemannian manifold, depends on the point at which the kernel it is centered, *e.g.* depends on the local geometry of **M** in a neighborhood of the observation.

$$f_{N,K}(\mathbf{Z}) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\theta_{\mathbf{Z}_i}(\mathbf{Z})}\frac{1}{h^n}K\left(\frac{\mathrm{D}(\mathbf{Z},\mathbf{Z}_i)}{h}\right) \qquad \theta_{\mathbf{P}}: \mathbf{Q} \rightarrow \theta_{\mathbf{P}}(\mathbf{Q}) = \frac{\mu_{\exp^*_{\mathbf{P}} g}}{\mu_{g_{\mathbf{P}}}}\left(\exp^{-1}_{\mathbf{P}}\mathbf{Q}\right)$$

- It is possible to ensure the integral is the same irrespective of where it is centered and make sure that the density function on **M** integrates to one by using the **volume density function**.
- If **Q** belongs to a normal neighborhood of **P**, then $\theta_{\mathbf{P}}(\mathbf{Q})$ is the density of the pullback of the volume measure on **M** to $T_{\mathbf{P}}\mathbf{M}$ with respect to the Lebesgue measure on $T_{\mathbf{P}}\mathbf{M}$ via the inverse exponential-map at point **P** .
  It gives an indication of the curvature of the Riemannian space.
  This is the same as the **square-root of the determinant of the metric-tensor** : ➡ $\theta_{\mathbf{P}}(\mathbf{Q}) = \left(\sqrt{|G_{\mathbf{P}}(y)|}\right)$
- The kernel estimator on Riemannian manifolds is consistent with standard kernel estimators on R and it converges at the same rate as the Euclidean kernel estimator.

Affine-Invariant Riemannian Metric

- An **Affine-Invariant** Riemannian metric can be deduced on the tensor manifold in terms of the **Fisher information matrix**.

$g_{ij} = g(E_i, E_j) = \langle E_i, E_j\rangle_{\mathbf{P}} = \frac{1}{2}\mathrm{tr}(\mathbf{P}^{-1}E_i\mathbf{P}^{-1}E_j)$ ➡ Affine-Invariant metric for the tensor manifold derived from the Fisher matrix.

$\mathrm{D}_a(\mathbf{P},\mathbf{Q}) = \sqrt{\frac{1}{2}\mathrm{tr}(\log^2(\mathbf{P}^{-\frac{1}{2}}\mathbf{Q}\mathbf{P}^{-\frac{1}{2}}))}$ ➡ Geodesic distance induced by the Affine-Invariant metric, derived from the Fisher matrix.

- Considering the normal coordinate system around **P** and the Ricci in this system and let y be the normal coordinates of **Q** :

$$\{\partial_i\}_{i=1,\dots,n} = \{E_i\}_{i=1,\dots,n} \qquad \boxed{\left(\sqrt{|G_{\mathbf{P}}(y)|}\right) \approx \left(1 - \frac{y^T\,\mathcal{R}\,y}{6}\right)} \qquad \forall i,j,k,l = 1,\dots,n$$

- The Riemannian curvature for the tensor manifold, derived from the Fisher matrix, and the classical Levi-Civita affine connection :

$$\mathrm{R}_{ijkl} = \mathrm{R}(E_i,E_j,E_k,E_l) = \frac{1}{4}\mathrm{tr}\left(E_j\mathbf{P}^{-1}E_i\mathbf{P}^{-1}E_k\mathbf{P}^{-1}E_l\mathbf{P}^{-1}\right) - \frac{1}{4}\mathrm{tr}\left(E_i\mathbf{P}^{-1}E_j\mathbf{P}^{-1}E_k\mathbf{P}^{-1}E_l\mathbf{P}^{-1}\right)$$

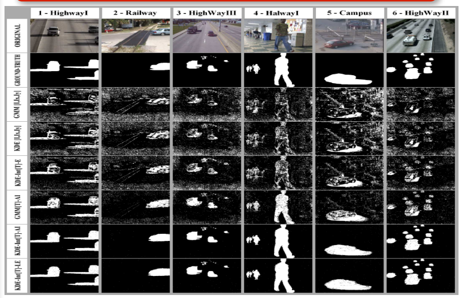
Log-Euclidean Riemannian Metric

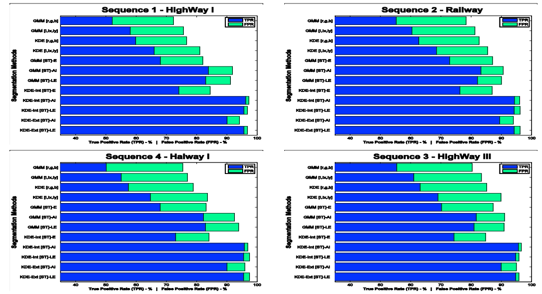
- The Log-Euclidean metric induces a space with a null curvature, while the theoretical properties are preserved.

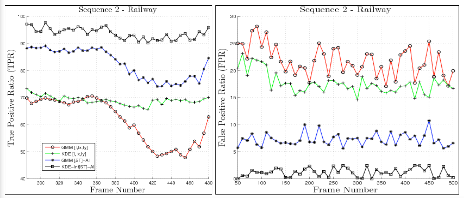
$\mathrm{D}_l(\mathbf{P},\mathbf{Q}) = \sqrt{\mathrm{tr}\left((\log(\mathbf{Q}) - \log(\mathbf{P}))^2\right)}$ ➡ The geodesic distance induced by the Log-Euclidean metric, is extremely simplified.

- Endowed with the Log-Euclidean metric the tensor space is **isomorphic**, **diffeomorphic** and **isometric** to the associated Euclidean space of **symmetric matrices**.
- Endowed with the Log-Euclidean metric, the tensor manifold is a **flat Riemannian space** (sectional curvature is null everywhere).
  When the Riemannian space is flat the volume density function is **unity everywhere**.
  The **isometry** implies that the determinant of the metric tensor is **unity everywhere**. ➡ $\left(\sqrt{|G_{\mathbf{P}}(y)|}\right) = 1$

Experimental Results







Methods	1-HighWayI TPR	1-HighWayI FPR	2-Railway TPR	2-Railway FPR	3-HighWayIII TPR	3-HighWayIII FPR	4-HalwayI TPR	4-HalwayI FPR	5-Campus TPR	5-Campus FPR	6-HighWayII TPR	6-HighWayII FPR
GMM [r, g, b]	52.10	20.25	55.23	23.10	55.40	24.95	50.10	25.30	48.50	34.48	50.63	37.14
GMM [I, Ix, Iy]	58.05	17.58	60.45	20.80	61.05	22.35	55.05	21.90	54.30	31.05	55.12	34.03
KDE [r, g, b]	59.83	16.85	62.70	19.95	63.15	22.00	57.45	21.40	55.60	30.13	58.25	32.58
KDE [I, Ix, Iy]	65.95	15.10	68.55	16.90	69.03	20.85	64.73	18.85	60.10	27.65	63.90	28.10
GMM [Tensor] - E	68.02	14.00	72.90	14.20	70.24	17.05	67.95	15.18	64.21	24.59	65.04	25.04
GMM [Tensor] - AI	83.90	07.95	83.25	07.38	81.70	09.35	82.27	10.25	74.94	14.01	73.81	15.20
GMM [Tensor] - LE	83.00	08.21	82.10	07.92	80.96	09.94	82.93	10.96	72.82	14.93	73.02	15.86
KDE-Int [Tensor] - E	74.10	10.36	76.30	10.65	74.35	10.46	73.05	11.03	69.04	15.08	70.13	18.83
KDE-Int [Tensor] - AI	96.25	01.02	94.35	01.74	95.65	00.95	95.78	01.12	87.90	05.52	87.25	07.95
KDE-Int [Tensor] - LE	95.64	01.17	94.23	01.96	94.75	01.08	95.53	01.95	87.14	05.71	86.97	07.13
KDE-Ext [Tensor] - AI	90.05	04.10	89.45	04.51	89.95	05.01	90.03	05.93	81.42	10.76	79.53	12.07
KDE-Ext [Tensor] - LE	95.64	01.17	94.23	01.96	94.75	01.08	95.53	01.95	87.14	05.71	86.97	07.13

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