SEMI-INTRINSIC MEAN SHIFT ON RIEMANNIAN MANIFOLDS

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Overview

Motivation :

- The mean shift (MS) on Euclidean spaces was extrinsically formulated to operate on general Riemannian manifolds (Ext-MS) . The mode seeking is performed on the tangent spaces, where the underlying curvature is not fully considered.
- The state-of-the-art method propose a intrinsic mean shift (IntGS-MS) designed to operate on two particular Riemannian manifolds, i.e. Grassmann and Stiefel manifolds (using manifold-dedicated density kernels).

Contribution :

- We propose a new paradigm to intrinsically reformulate the MS on general Riemannian (SInt-MS) manifolds by embedding the Riemannian manifold into a Reproducing Kernel Hilbert Space using a general Riemannian kernel function, i.e. heat kernel.
- The key issue is that when the data is implicitly mapped from the manifold to the Hilbert space, the structure and curvature of the manifold is taken into account (i.e. exploits the underlying information of the data).
- The inherent optimization is then performed on the embedded space.

Kernel Mean Shift

- The manifold data is implicitly mapped to an enlarged feature $\mathcal{H}$ space by the mapping function $\mathbf{z} = \phi(\mathbf{Z})$
- In the feature space the density estimator at the point $\mathbf{z} \in \mathcal{H}$

$$f_{\mathcal{H}}(\mathbf{z}) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{h_i^d}k\left(\|\frac{\mathbf{z}-\phi(\mathbf{Z}_i)}{h_i}\|^2\right) \qquad \mathbf{Z}_i \in \mathcal{M}$$

$$K(\mathbf{Z},\mathbf{Z}') = \phi(\mathbf{Z})^T\phi(\mathbf{Z}') \qquad K: \mathcal{M}\times\mathcal{M} \rightarrow \Re$$

Input: 1 - $\mathbf{K}$ computed using Heat Kernel on Riemannian manifolds (Section 4)
2 - Bandwidth selection parameter k

Calculate the bandwidths h_i as the k^{th} smallest distance from the point using Eq. 4 with $\mathbf{K}$ and $d = \text{rank}(\mathbf{K})$

for *All data points* $i = 1, ..., N$ **do** $\qquad \phi(\mathbf{Z}_i) = \mathbf{\Phi}\mathbf{e}_i \qquad \mathbf{z} = \mathbf{\Phi}\alpha_{\mathbf{z}}$

a) - Let $\alpha_{\mathbf{z}_i} = \mathbf{e}_i$

b) - Repeat until convergence (with $D' = \alpha_{\mathbf{z}_i}^T\mathbf{K}\alpha_{\mathbf{z}_i} + \mathbf{e}_j^T\mathbf{K}\mathbf{e}_j - 2\alpha_{\mathbf{z}_i}^T\mathbf{K}\mathbf{e}_j$)

$$\bar{\alpha}_i = \left(\sum_{j=1}^{N}\frac{\mathbf{e}_j}{h_j^{d+2}}g\left(\frac{D'}{h_j^2}\right)\right)\left(\sum_{j=1}^{N}\frac{1}{h_j^{d+2}}g\left(\frac{D'}{h_j^2}\right)\right)^{-1} \qquad (5)$$

c) - Group the points $\bar{\alpha}_{\mathbf{z}_i}$ and $\bar{\alpha}_{\mathbf{z}_j}$, $i, j = 1, ..., N$ satisfying $\bar{\alpha}_i^T\mathbf{K}\bar{\alpha}_i + \bar{\alpha}_j^T\mathbf{K}\bar{\alpha}_j - 2\bar{\alpha}_i^T\mathbf{K}\bar{\alpha}_j = 0$.

$$\mathbf{\Phi} = [\phi(\mathbf{Z}_1) \;\; \phi(\mathbf{Z}_2) \;\; \ldots \;\; \phi(\mathbf{Z}_N)] \qquad \mathbf{K} = \mathbf{\Phi}^T\mathbf{\Phi}$$

Mercer Kernel on Riemannian Manifolds - Heat Kernel

- A Riemannian manifold can be embedded into a Hilbert space using the heat kernel (it was proved that the heat kernel can define a Mercer Kernel while respect the Riemannian geometry).
- The Laplace operator on a Riemannian Manifold is defined as

$$\mathcal{L} = \nabla^*\nabla + \mathcal{Q} = -g^{\mu\nu}\nabla_\mu\nabla_\nu + \mathcal{Q}$$

- The kernel of that operator satisfies the heat equation and is called heat kernel (expansions of eigenfunctions of the Laplace)

$$K_t(t|\mathbf{Z},\mathbf{Z}') = U_{\mathcal{L}}(t|\mathbf{Z},\mathbf{Z}') = \sum_{l=1}^{\infty} e^{-t\lambda_l}\varphi_l(\mathbf{Z}) \otimes \bar{\varphi}_l(\mathbf{Z}')$$

- We seek inspiration from physics adopting an asymptotic expansion and tractable solution of the heat kernel applicable for generic Riemannian manifolds $t = \sigma^2$

$$= (4\pi t)^{-\frac{n}{2}}\Delta(\mathbf{Z},\mathbf{Z}')^{\frac{1}{2}}\exp\left(-\frac{W(\mathbf{Z},\mathbf{Z}')}{2t}\right)\mathcal{P}(\mathbf{Z},\mathbf{Z}')\Omega(t|\mathbf{Z},\mathbf{Z}')$$

$W(\mathbf{Z},\mathbf{Z}')$ World function $\qquad \Delta(\mathbf{Z},\mathbf{Z}')$ Van Vleck-Morette determinant

$\mathcal{P}(\mathbf{Z},\mathbf{Z}')$ Parallel transport operator $\qquad \Omega(t|\mathbf{Z},\mathbf{Z}')$ Transport function

Experimental Evaluation

$\mathcal{G}$ m	k	Performance Ext-Ms	SInt-Ms	$\mathcal{V}$ m	k	Performance Ext-Ms	SInt-Ms	SO_3 r	-	Performance Ext-Ms	SInt-Ms	$\mathcal{S}^+$ d	d	Performance Ext-Ms	SInt-Ms
A ↦ Classes = 4 | Points per Class = 100															
3	1	51.23	63.42	3	1	64.83	84.05	1	-	55.53	78.23	3	3	57.68	85.64
5	3	67.41	78.50	3	3	71.15	86.25	2	-	66.78	86.86	4	4	66.46	91.04
5	4	46.80	61.91	5	3	77.10	91.76	3	-	59.45	81.35	5	5	65.71	92.56
10	4	53.27	65.34	10	1	74.91	90.35	4	-	61.59	82.34	6	6	65.75	90.84
20	4	50.05	64.08	50	1	70.03	87.50	5	-	57.50	80.29	7	7	61.28	88.39
B ↦ Classes = 4 | Points per Class = 200															
3	1	49.51	72.25	3	1	60.25	84.24	1	-	52.38	82.74	3	3	53.81	87.95
5	3	64.28	82.39	3	3	66.12	87.05	2	-	62.70	89.20	4	4	61.91	92.63
5	4	45.23	69.72	5	3	69.86	91.41	3	-	55.04	85.06	5	5	59.72	92.73
10	4	51.08	73.01	10	1	67.05	90.25	4	-	56.41	86.13	6	6	59.30	92.68
20	4	49.16	72.89	50	1	65.41	88.95	5	-	54.78	85.42	7	7	57.59	91.68
C ↦ Classes = 8 | Points per Class = 100															
3	1	46.85	69.85	3	1	57.85	80.53	1	-	49.85	79.69	3	3	51.35	84.61
5	3	60.05	78.05	3	3	60.25	86.01	2	-	57.92	86.53	4	4	56.45	90.47
5	4	42.91	67.91	5	3	65.12	84.25	3	-	51.31	80.57	5	5	55.81	86.91
10	4	47.35	69.35	10	1	63.51	82.42	4	-	52.93	80.34	6	6	55.72	85.90
20	4	45.19	69.19	50	1	60.39	83.05	5	-	50.29	80.62	7	7	52.84	86.35
D ↦ Classes = 8 | Points per Class = 200															
3	1	44.05	71.05	3	1	53.83	80.05	1	-	46.44	80.05	3	3	47.63	84.51
5	3	57.35	78.35	3	3	55.15	83.10	2	-	53.75	85.25	4	4	51.95	88.60
5	4	41.03	70.03	5	3	58.10	81.31	3	-	47.06	80.17	5	5	50.08	85.24
10	4	45.15	67.15	10	1	59.91	85.65	4	-	50.03	80.93	6	6	52.47	87.95
20	4	44.10	70.10	50	1	54.03	79.19	5	-	46.82	79.14	7	7	47.79	83.56
Average Improvement : Δ = (SInt-MS) - (Ext-MS)															
$\Delta\mathcal{G}$ = + 20 %				$\Delta\mathcal{V}$ = + 22 %				ΔSO_3 = + 28 %				$\Delta\mathcal{S}^+$ = + 32 %			

Table 1. Clustering rates (%) of extrinsic (Ext-MS) and semi-intrinsic (SInt-MS) meanshift on Riemannian manifolds in the case of simulations on synthetic data.

We present results for the manifolds :

Grassmann manifold $(\mathcal{G}_{k,m-k})$

Stiefel manifold $(\mathcal{V}_{k,m})$

Special Orthogonal Group (SO_3)

Symmetric Positive-Definite $(\mathcal{S}_d^+)$

$\mathcal{G}$ Scales	Bins	m	k	Performance Ext-Ms	SInt-Ms	$\mathcal{V}$ Scales	Bins	m	k	Performance Ext-Ms	SInt-Ms	$\mathcal{S}^+$ d	d	Performance Ext-Ms	SInt-Ms
E ↦ Classes = 3 | Points per Class = 250															
2	32	32	4	68.31	88.46	2	32	128	1	69.83	91.05	5	5	63.75	85.44
2	64	64	4	76.95	95.40	2	64	256	1	76.15	93.22	7	7	62.02	84.22
3	32	32	6	72.28	91.26	3	32	192	1	81.10	97.55	9	9	58.50	82.45
F ↦ Classes = 3 | Points per Class = 300															
2	32	32	4	63.98	91.55	2	32	128	1	65.25	91.24	5	5	55.12	91.84
2	64	64	4	73.55	97.53	2	64	256	1	71.12	94.05	7	7	53.29	89.50
3	32	32	6	67.35	94.68	3	32	192	1	73.86	91.41	9	9	58.16	86.11
G ↦ Classes = 4 | Points per Class = 250															
2	32	32	4	61.47	88.42	2	32	128	1	62.85	87.53	5	5	52.72	83.65
2	64	64	4	68.60	95.09	2	64	256	1	65.25	93.01	7	7	54.97	80.21
3	32	32	6	63.84	89.62	3	32	192	1	70.12	91.25	9	9	52.03	81.42
H ↦ Classes = 4 | Points per Class = 300															
2	32	32	4	57.98	88.67	2	32	128	1	58.83	87.05	5	5	48.61	82.18
2	64	64	4	64.80	93.58	2	64	256	1	60.15	90.10	7	7	49.24	86.24
3	32	32	6	61.08	88.63	3	32	192	1	63.10	88.31	9	9	45.28	81.66
Average Improvement : Δ = (SInt-MS) - (Ext-MS)															
$\Delta\mathcal{G}$ = + 25 %						$\Delta\mathcal{V}$ = + 23 %						$\Delta\mathcal{S}^+$ = + 30 %			

Table 2. Clustering rates (%) of extrinsic (Ext-MS) and semi-intrinsic (SInt-MS) meanshift on Riemannian manifolds on object categorization for selected objects on the data set ETH-80[30].

The clustering accuracy was evaluated on synthetic data (Table 1) as well as on real data (Table 2) comparing the proposed mean shift (SInt-MS) with the extrinsic counterpart (Ext-MS) [2,13] (serve mainly as a proof of concept).

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